A Numerical Examination of the Optical Properties of Thin Metallic Plates.

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The optical properties of thin metallic plates have been investigated by a number of physicists.* One of the earliest workers in this field was MacCullagh.† He predicted from theory and verified by experiment that if light incident on a gold leaf were plane polarised the transmitted beam would be elliptically polarised. With the improvement in experimental methods since MacCullagh's day, and the gradual removal of obscurities from the theory of metallic reflection and transmission, we now expect much more than a mere general agreement between theory and experiment. We look for an almost exact *numerical* coincidence.‡

The condition of the reflected or transmitted beam is precisely described by means of two quantities—the ellipticity and the difference of phase between the components of the light polarised perpendicular and parallel to the plane of incidence. The object of the present paper is to obtain convenient formulæ for these quantities and to compare them with the results of experiments, selecting the most careful and the most recent that are available. We shall admit into our theory no principle that has not found general acceptance,§ and shall thus be enabled to decide whether such principles are sufficient to colligate all the facts. If they fail in this, it will behove us to look for new principles, or, by a scrutiny of our so-called "facts," to indicate in what way experimental errors have brought about an apparent conflict between fact and theory.

Suppose that a ray of light is incident at an angle ϕ on a metal plate. If the plane of xy be taken as that of incidence, then an incident wave of unit amplitude will be represented by a vector of the form $e^{ip[t-\mathrm{V}^{-1}(x\cos\phi+y\sin\phi)]}$, where $p/2\pi$ is the frequency, t the time, V the velocity of light in air. This incident wave will give rise to a reflected wave represented by $re^{ip[t+\mathrm{V}^{-1}(x\cos\phi-y\sin\phi)]}$ and a refracted wave $se^{ip[t-\mathrm{V}^{-1}(bx+y\sin\phi)]}$. The dynamical equation requires that $b^2+\sin^2\phi=\mu^2$, where μ is a complex of the form μ_1-ia. Here μ_1 is the "refractive index" of the metal, *i.e.*, the ratio of the velocity of light in air to that in the metal, and a is the coefficient of absorption.

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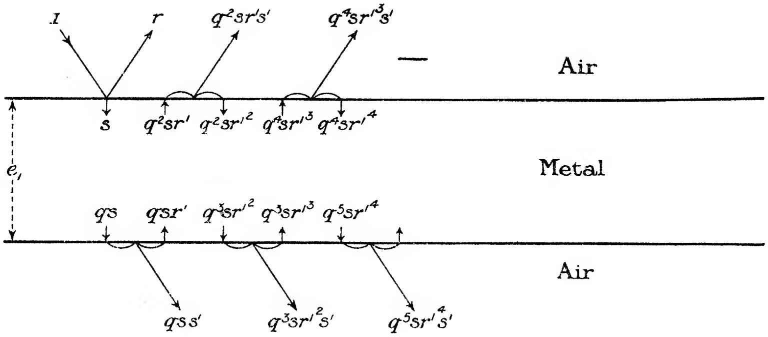
* See references in Mascart, 'Traité d'Optique,' vol. 2, p. 550, and Winkelmann, 'Handbuch der Physik,' vol. 6 (2), p. 1311. Neither of these, however, mentions the work of Conroy, 'Roy. Soc. Proc.,' vol. 31, 1881, p. 486.

† 'Roy. Irish Acad. Proc.,' vol. 1, p. 27, and 'Collected Works,' p. 82.

‡ See, *e.g.*, 'Roy. Soc. Proc.,' A, vol. 77, 1906, p. 211.

§ Of recent investigations the most elaborate is that of M. Meslin, 'Ann. de Chimie et de Phys.,' 1890, pp. 56—177. After giving the results of experiments, M. Meslin obtains, empirically, a formula for the difference of phase produced by transmission through a plate. He then makes an ingenious attempt to deduce this formula from theory. His argument, however, contains too many special hypotheses and assumptions to be final.



The refracted wave $se^{ip[t-\mathrm{V}^{-1}(bx+y\sin\phi)]}$, which we shall refer to briefly as the wave s, will, after traversing the plate, become qs. When it reaches the bounding surface it will be reflected and refracted. In this way waves will surge to and fro across the plate, while an infinite series of waves will return into the first medium (air) and constitute the reflected beam, and another series will pass into the third medium (air) and constitute the transmitted beam.

We shall denote by r' and s' the quantities corresponding to r and s when we are considering a passage from metal to air instead of from air to metal. With this notation the reflected beam is represented by

$$r+q^2ss'r'+q^4ss'r'^2+\ldots=r+\frac{q^2ss'r'}{1-q^2r'^2},$$

and the transmitted beam by

$$qss'+q^3ss'r'^2+q^5ss'r'^4+\ldots=\frac{qss'}{1-q^2r'^2}.$$

The values of the quantities r and r', s and s' are given by the theory of reflection and refraction, and in this way we can easily prove that $r+r'=0$, and $ss'=1-r^2=1-r'^2$. These results can also be obtained by means of Stokes' principle of reversion, or by noting that when the thickness of the metal is indefinitely diminished and q becomes unity, then all the light is transmitted and none reflected.

Making use of these relations, the reflected beam becomes $r(1-q^2)/(1-q^2r^2)$

and the transmitted beam $q(1-r^2)/(1-q^2r^2)$. It thus appears that everything will be determined when q and r are obtained.

The first refracted wave was $se^{ip[t-\mathrm{V}^{-1}(bx+y\sin\phi)]}$, where $b^2+\sin^2\phi=\mu^2$. Hence, if e_1 be the thickness of the plate, we have $q=e^{-ipbe_1\mathrm{V}^{-1}}=e^{-i2\pi be_1\lambda^{-1}}$, where λ is the wave-length of the incident light. Putting $\mu=\mu_1-ia=\mathrm{M}e^{-ia}$, we have $\mathrm{M}^2=\mu_1{}^2+a^2$, $\tan\alpha=a/\mu_1$, $b=+(\mu^2-\sin^2\phi)^{\frac{1}{2}}=me^{-i\chi}$, where $m^4=\mathrm{M}^4+\sin^4\phi-2\mathrm{M}^2\cos 2\alpha\sin^2\phi$, and $\tan 2\chi=\mathrm{M}^2\sin 2\alpha/(\mathrm{M}^2\cos 2\alpha-\sin^2\phi)$. We thus get $q^2=\mathrm{Q}e^{-i\psi}$, where $\mathrm{Q}=e^{-4\pi e_1\lambda^{-1}m\sin\chi}$ and $\psi=4\pi e_1\lambda^{-1}m\cos\chi$. These relations suffice to determine Q in terms of the optical constants of the metal and the thickness of the plate. The quantity r is also a complex. Its value is given by the theory of reflection and proves to be dependent on the position of the plane of polarisation of the incident light. If we regard the transition from one medium to the other to be abrupt, then, when the incident light is polarised perpendicularly to the plane of incidence, r is given by Fresnel's formula

$$r_1=\frac{\mu\cos\phi-\cos\phi'}{\mu\cos\phi+\cos\phi'}=\frac{\tan(\phi-\phi')}{\tan(\phi+\phi')}.$$

For light polarised parallel to the plane of incidence, we have the corresponding formula

$$r_2=\frac{\cos\phi-\mu\cos\phi'}{\cos\phi+\mu\cos\phi'}=\frac{-\sin(\phi-\phi')}{\sin(\phi+\phi')}.$$

We have $\qquad \sin\phi=(\mu_1-ia)\sin\phi'$,

and putting $\qquad \cos\phi'=ce^{-iu}=(1-\mathrm{M}^{-2}\sin^2\phi e^{2i\alpha})^{\frac{1}{2}}$,

we get*

$$r_1=\mathrm{R}_1e^{i\theta_1},\quad\text{where}\quad \mathrm{R}_1{}^2=\frac{1-x_1}{1+x_1}\quad\text{and}\quad x_1=\frac{2\mathrm{M}c\cos\phi\cos(\alpha-u)}{\mathrm{M}^2\cos^2\phi+c^2},$$

while $$\tan\theta_1=\frac{-2\mathrm{M}c\cos\phi\sin(\alpha-u)}{\mathrm{M}^2\cos^2\phi-c^2}.$$

Similarly

$$r_2=\mathrm{R}_2e^{i\theta_2},\qquad \mathrm{R}_2{}^2=\frac{1-x^2}{1+x^2},\qquad x_2=\frac{2\mathrm{M}c\cos\phi\cos(\alpha+u)}{\mathrm{M}^2c^2+\cos^2\phi},$$

$$\tan\theta_2=\frac{-2\mathrm{M}\cos\phi\sin(\alpha+u)}{\mathrm{M}^2c^2-\cos^2\phi}.$$

If the incident wave be represented by 1, then the reflected beam is similarly represented by $\mathrm{R}'e^{i\theta'}$, where R′ is the amplitude and θ' the change of phase produced by reflection. We have

$$\mathrm{R}'e^{i\theta'}=\frac{r(1-q^2)}{1-q^2r^2}=\frac{\mathrm{R}e^{i\theta}(1-\mathrm{Q}e^{-i\psi})}{1-\mathrm{Q}\mathrm{R}^2e^{i(2\theta-\psi)}}.$$

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* 'Roy. Soc. Proc.,' A, vol. 77, 1906, p. 212.

We thus get

$$R'^2 = R^2 \frac{1+Q^2-2Q\cos\psi}{1+Q^2R^4-2QR^2\cos(2\theta-\psi)}$$

$$= R^2 \frac{1+Q^2-2Q\cos\psi}{1+Q^2R^4-2QR^2\cos(\psi+\gamma)},$$

where $2\theta = 360^\circ-\gamma$.

This determines the intensity of the reflected beam. When the thickness of the plate is indefinitely small $Q = 1$ and $\psi = 0$. Hence $R'^2 = 0$, so that there is no reflected beam. As the thickness increases, Q approaches the limit zero, so that R'^2 approaches the limit R^2—as is to be expected. It may be interesting to observe, however, that the intensity of the reflected light may be greater for a thin plate than for a very thick one. We have $R' = R$ when

$$1+Q^2-2Q\cos\psi = 1+Q^2R^4-2QR^2\cos(\psi+\gamma),$$

$$\cos\psi-R^2\cos(\psi+\gamma) = \tfrac{1}{2}Q(1-R^4),$$

$$\sin(\psi-\gamma') = \frac{Q(1-R^4)}{2(1-2R^2\cos\gamma+R^4)^{\frac{1}{2}}},$$

where

$$\tan\gamma' = \frac{1-R^2\cos\gamma}{R^2\sin\gamma}.$$

Consider the case of direct incidence. We then have $\phi = 0$, $\psi = 4\pi e_1\lambda^{-1}\mu_1$, $Q = e^{-4\pi c_1\lambda^{-1}a} = e^{-c_1\psi}$, where $c_1 = \tan\alpha$. The equation to determine ψ thus becomes

$$\sin(\psi+\gamma') = \frac{e^{-c_1\psi}(1-R^4)}{2(1-2R^2\cos\gamma+R^4)^{\frac{1}{2}}}.$$

For the silver and gold, to be considered later, c_1 is large, so that when ψ is not very small, the quantity $e^{-c_1\psi}$ will be very small, and we shall have $\sin(\psi+\gamma') = 0$ very approximately. This makes $\psi = \pi-\gamma'$. In the case of silver this makes $\frac{e^{-c_1\psi}(1-R^4)}{2(1-2R^2\cos\gamma+R^4)^{\frac{1}{2}}} = 8\times 10^{-25}$. This is quite negligible, so that our approximation is sufficiently accurate for practical purposes.

Remembering that $e_1/\lambda = \psi/4\pi\mu_1$, we can thus easily find the thickness that gives $R' = R$. For the metals discussed later this gives $e_1/\lambda = 1{\cdot}136$ in the case of silver, and $e_1/\lambda = 0{\cdot}5343$ in the case of gold. If the thickness is rather greater than this, we have R' greater than R, but as the thickness increases, R' once more approaches to equality with R. There will then be some thickness that makes R' a maximum, and it is not difficult to determine this thickness.

We have $\frac{R'^2}{R^2} = \frac{1+Q^2-2Q\cos\psi}{1+Q^2R^4-2QR^2\cos(\psi+\gamma)}$, and R'^2 is to be made a maximum, it being observed that Q and ψ are functions of the thickness. Putting

$$\frac{e_1}{\lambda} = t_1; \quad Q = e^{-c_1\psi}; \quad \frac{dQ}{dt_1} = -c_1 Q \frac{d\psi}{dt_1}; \quad t_1 = \frac{\psi}{4\pi\mu_1};$$

we get, by equating dR'^2/dt_1 to zero,

$$[1+Q^2R^4-2QR^2\cos(\psi+\gamma)]\left[Q\frac{dQ}{dt_1}-\cos\psi\frac{dQ}{dt_1}+Q\sin\psi\frac{d\psi}{dt_1}\right]$$

$$= [1+Q^2-2Q\cos\psi]\left[R^4Q\frac{dQ}{dt_1}-R^2\cos(\psi+\gamma)\frac{dQ}{dt_1}+QR^2\sin(\psi+\gamma)\frac{d\psi}{dt_1}\right],$$

i.e. $[1+Q^2R^4-2QR^2\cos(\psi+\gamma)][-Q+\cos\psi+c_1^{-1}\sin\psi]$

$$= [1+Q^2-2Q\cos\psi][-R^4Q+R^2\cos(\psi+\gamma)+R^2c_1^{-1}\sin(\psi+\gamma)],$$

i.e., $[\sin(\psi+\alpha)-R^2\sin(\psi+\gamma+\alpha)]+Q[2R^2\sin\gamma\cos\alpha-(1-R^4)\sin\alpha]$

$$+Q^2R^2[R^2\sin(\psi-\alpha)-\sin(\psi+\gamma-\alpha)] = 0.$$

As Q is small, this can easily be solved by approximations. The first approximation gives

$$\sin(\psi+\alpha)-R^2\sin(\psi+\gamma+\alpha) = 0,$$

i.e., $\cos(\psi+\gamma'+\alpha) = 0, \qquad \gamma = \tfrac{3}{2}\pi-\alpha-\gamma'.$

For silver, this makes $Q = e^{-c_1\psi} = 6{\cdot}6 \times 10^{-24}$, which is so small that the first approximation is sufficiently accurate. The thickness that gives the greatest intensity of reflected light is determined by $e_1/\lambda = 1{\cdot}157$ in the case of silver, and $e_1/\lambda = 0{\cdot}5586$ in the case of gold. In both cases, however, the maximum value of R'^2 is *very* slightly greater than R^2.

The following table shows how the intensity of the reflected light depends on the thickness of the plate in the case of gold and silver. The intensity, of course, depends on the values of the optical constants α and μ_1, and these in turn vary with the colour of the incident light. From Drude's experiments[*] with well cleaned silver in sodium light, he calculated that $a = 3{\cdot}67$, and $\mu_1 = 0{\cdot}18$, and from similar experiments by Conroy,[†] with gold in red light, we find that $a = 3{\cdot}204$, and $\mu_1 = 0{\cdot}365$[‡].

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[*] 'Wied. Ann.,' vol. 39, p. 481.

[†] 'Roy. Soc. Proc.,' 1881, p. 486.

[‡] These constants are deduced from observations of the Principal Incidence and the Principal Azimuth. It has often been observed that these angles vary considerably with different conditions of the reflecting surface. That this is to be expected is apparent if we regard the transition from one medium to the other as gradual and not abrupt. The Principal Incidence and Principal Azimuth depend on the nature of the layer of transition, and when the matter is examined carefully it is found that the neglect of the transition layer may lead to considerable errors in the values of the optical constants. See 'Roy. Soc. Proc.,' 1906, p. 211, and *post*, p. 330, note †. For the present we shall take the value derived from the experiments of Drude and Conroy on the hypothesis of an abrupt transition as forming a sufficiently accurate first approximation to the truth.

e_1/λ.	Indef. small.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.	∞ .
Silver R'^2............	0	0·0020	0·0610	0·1632	0·4301	0·8260	0·9383	0·9516	0·9517
Gold R'^2	0	0·0012	0·0289	0·1012	0·2922	0·6803	0·8504	0·8794	0·8796

These results are represented graphically in fig. 1 below.

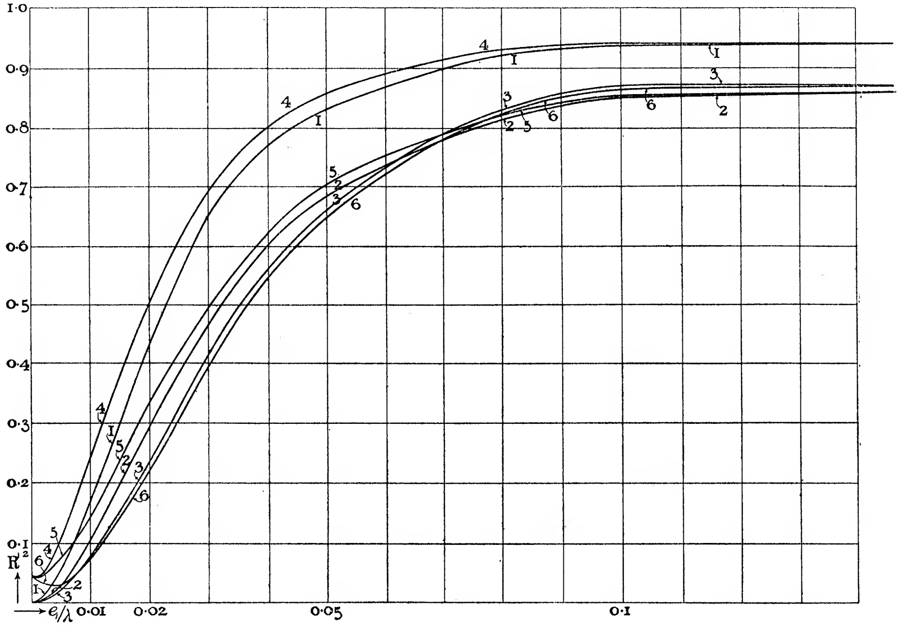


Fig. 1.

Curve 1 represents the silver and Curve 2 the gold. It will be seen that in both cases there is very little change in the intensity after a thickness of one-tenth of the wave-length has been reached. For that thickness the value of R'^2 is nearly 99 per cent. of that for a thick plate.

The formula for $R'e^{i\theta'}$ on p. 298 also enables us to calculate θ', the change of phase produced by reflection. We have $\theta' = \theta + \Phi' - \Phi$, where

$$\tan \Phi = \frac{-QR^2 \sin(2\theta-\psi)}{1-QR^2\cos(2\theta-\psi)} = \frac{QR^2\sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)},$$

and Φ' is obtained from Φ by putting $R = 1$ and $\mu = 0$. As the thickness increases Q diminishes rapidly, so that Φ and Φ' approach the limit zero, and θ' approaches asymptotically the value θ, as is to be expected. In doing this, however, θ' may pass through a maximum and be greater than θ, so that the change of phase produced by a thin plate may be greater than that produced by a thick plate—as we have already seen is the case with the intensity.

We have $\theta' = \theta$ when $\Phi' = \Phi$, *i.e.*,

$$\frac{QR^2 \sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)} = \frac{Q\sin\psi}{1-Q\cos\psi},$$

$$\cos\psi\,[R^2\sin\gamma] - \sin\psi\,[1-R^2\cos\gamma] = QR^2\sin\gamma,$$

$$\cos(\psi+\gamma') = \frac{QR^2\sin\gamma}{(1-2R^2\cos\gamma+R^4)^{\frac{1}{2}}} = e^{-c_1\psi}\frac{R^2\sin\gamma}{(1-2R^2\cos\gamma+R^4)^{\frac{1}{2}}}.$$

This is satisfied, very approximately, by $\psi+\gamma' = \frac{1}{2}\pi$. For silver this corresponds to $e_1/\lambda = 0{\cdot}441$, and for gold $e_1/\lambda = 0{\cdot}192$.

To make θ' a maximum we must have

$$[\cos\psi - c_1\sin\psi - Q]\,[1-2QR^2\cos(\psi+\gamma)+Q^2R^4]$$
$$= R^2[1-2Q\cos\psi+Q^2]\,[\cos(\psi+\gamma)-c_1\sin(\psi+\gamma)-QR^2],$$

i.e., $$[\cos(\psi+\alpha)-R^2\cos(\psi+\alpha+\gamma)] - Q\,[(1-R^4)\cos\alpha+2R^2\sin\gamma\sin\alpha]$$
$$-2Q^2R^2\sin\tfrac{1}{2}\gamma\sin(\psi-\alpha+\tfrac{1}{2}V) = 0.$$

As a first approximation $\cos(\psi+\alpha) - R^2\cos(\psi+\alpha+\gamma) = 0$, *i.e.*, $\sin(\psi+\alpha-\gamma')=0$, so that $\psi+\alpha+\gamma'=\pi$. For both silver and gold this will be found to make ψ nearly unity, so that Q is very small, and the first approximation is sufficiently close. The maximum value of θ' occurs when $\psi+\gamma'=\pi-\alpha$ and we have seen that $\theta'=\theta$ when $\psi+\gamma'=\frac{1}{2}\pi$. α is less than $\frac{1}{2}\pi$, so that $\pi-\alpha$ is greater than $\frac{1}{2}\pi$; but in the case of the silver and gold, already discussed, it is not very much greater, so that the maximum is reached soon after θ' is equal to θ. For silver the maximum occurs when $e_1/\lambda=0{\cdot}455$ and for gold when $e_1/\lambda=0{\cdot}216$.

The following table gives the value of θ', *expressed as a fraction of the half wave-length.*

e_1/λ.	Indef. small.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.	∞ .
Silver θ'	0·5288	0·5417	0·5905	0·6410	0·7171	0·8061	0·8288	0·8312	0·8312
Gold θ'...............	0·5658	0·5771	0·6095	0·6473	0·7050	0·7832	0·8072	0·8095	0·8095

These results are represented graphically in fig. 2 below, Curve 1 corresponding to the silver and Curve 2 to the gold.

Perhaps the most striking thing about these results is the large percentage of the change of phase due to a thick plate which is produced by a very thin plate. In the one case 64 per cent. and in the other 70 per cent. of the total change is produced by an *indefinitely* thin plate. It must be borne in mind, however, that the intensity of the reflected light for such thicknesses is vanishingly small, so that it would not be practicable to measure the

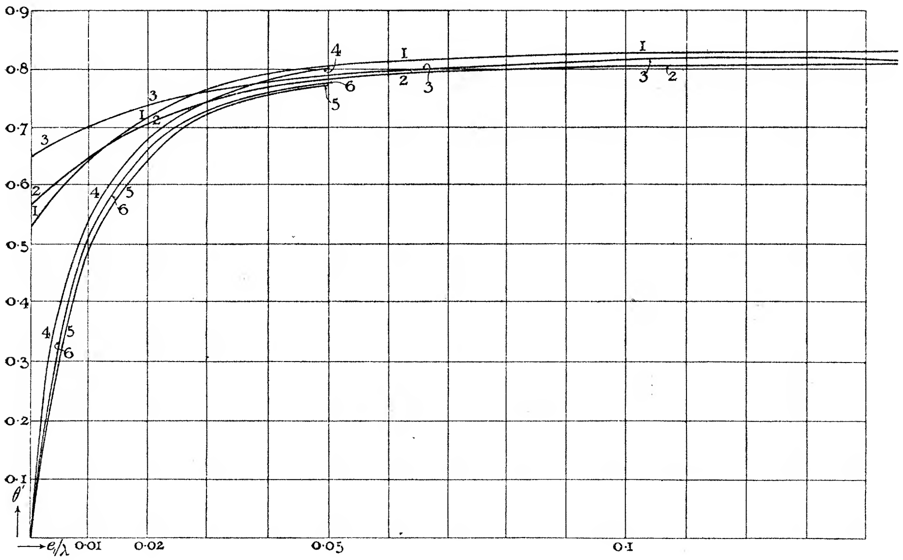


FIG. 2.

change of phase even if such thin plates were available. And in any case we should not be justified in pushing our analysis to the extreme of "vanishing" thinness, for when we approach molecular dimensions the fundamental hypothesis that the granular structure of reality can be replaced for optical purposes by a *continuum* ceases to be tenable.

Few accurate measurements seem to have been made on the intensity and change of phase in the case of reflection from a thin metal plate with air surrounding it. *Extreme* tenuity can be obtained only by depositing the metal chemically or electrically on some other material, such as glass. This, as we shall see later, completely alters the character of the reflected light when the thickness of the metal is small; but before this matter is discussed it will be convenient to investigate the nature of the transmitted beam for a metal plate surrounded by air.

For the transmitted beam we have

$$R'e^{i\theta'} = \frac{q(1-r^2)}{1-q^2r^2},$$

so that the intensity is

$$R'^2 = \frac{Q(1+R^4-2R^2\cos 2\theta)}{1+Q^2R-2QR^2\cos(2\theta-\psi)} = \frac{Q(1+R^4-2R^2\cos\gamma)}{1+Q^2R^4-2QR^2\cos(\psi+\gamma)}.$$

As the thickness of the plate increases Q diminishes rapidly, so that the intensity of the transmitted light diminishes rapidly to zero. The following table gives the intensity for the gold and silver already discussed:—

e_1/λ.	Indef. small.	0 ·001.	0 ·005.	0 ·01.	0 ·02.	0 ·05.	0 ·1.	0 ·2.	∞ .
Silver R'^2............	1	0 ·9608	0 ·8705	0 ·7418	0 ·4644	0 ·1	0 ·0094	0 ·0001	0
Gold R'^2	1	0 ·9605	0 ·9050	0 ·7820	0 ·5423	0 ·1541	0 ·0201	0 ·0007	0

These results are represented graphically in fig. 3.

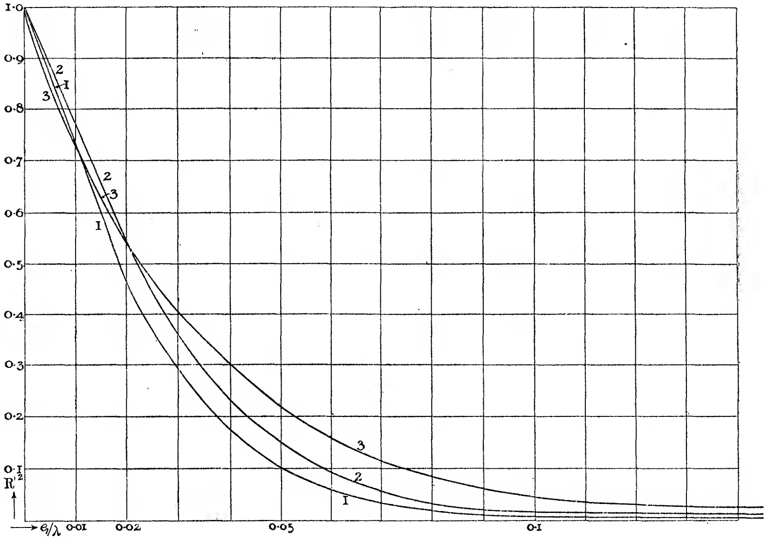


Fig. 3.

In this figure Curve 1 corresponds to the silver and Curve 2 to the gold. A silver plate whose thickness is one-tenth of the wave-length is practically opaque.

The change of phase is given by $\theta' = \Phi'' - \Phi - \frac{1}{2}\psi$, where Φ has the same value as on p. 301, and Φ'' is obtained from Φ by putting $Q=1$ and $\psi=0$. As the thickness increases Φ rapidly diminishes, so that the change of phase is soon represented by $\Phi'' = \frac{1}{2}\psi$. Φ'' is a constant, while $\frac{1}{2}\psi$ increases with the thickness, so that, after a certain thickness has been reached, θ' diminishes steadily and ultimately becomes negative. (It must be remembered that meanwhile Q diminishes rapidly and the intensity of the transmitted light becomes vanishingly small.) Φ diminishes at first more rapidly than $\frac{1}{2}\psi$ increases, so that θ' will begin by increasing and will afterwards diminish. It must, therefore, pass through a maximum and we proceed to determine the thickness of metal that will produce this maximum change of phase. Φ'' is a constant, so that we have to make $\Phi + \frac{1}{2}\psi$ a minimum. Remembering that $\tan \Phi = \dfrac{QR^2 \sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)}$ and that $Q = e^{-c_1\psi}$, we get, to determine the maximum of θ', the equation

$$1-2c_1QR^2\sin(\psi+\gamma)-Q^2R^4 = 0.$$

I.e.,

$$\sin(\psi+\gamma) = \frac{1}{2c_1}\left[\frac{e^{c_1\psi}}{R^2}-\frac{R^2}{e^{c_1\psi}}\right].$$

This is easily solved by successive approximations, and yields $e_1/\lambda = 0{\cdot}078$ in the case of silver, and $e_1/\lambda = 0{\cdot}068$ in the case of gold.

The value of θ', expressed as a fraction of the half wave-length, is given in the following table, and the results are represented graphically in fig. 4 below. As before, Curve 1 corresponds to the silver and Curve 2 to the gold.

e_1/λ.	Indef. small.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.
Silver θ' …………	0	0·0123	0·0642	0·1111	0·1872	0·2707	0·2787	0·2456
Gold θ' …………	0	0·0111	0·0432	0·0808	0·1374	0·2055	0·2018	0·1277

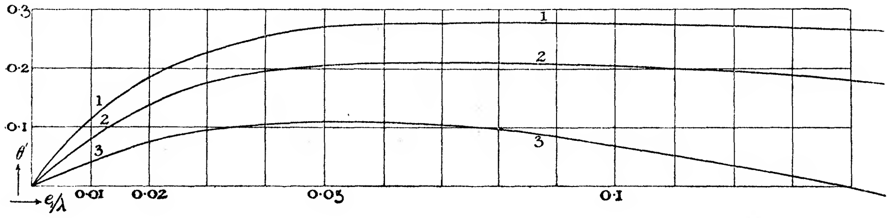
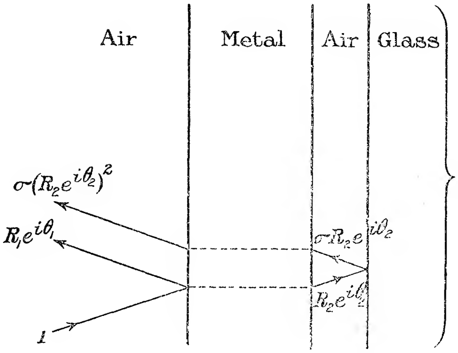


FIG. 4.

We shall now consider to what extent the results already obtained as to the intensity and change of phase of the reflected light are modified when the metal is deposited on glass. The second reflection and refraction in the investigation of p. 297 now takes place at a passage from metal to glass instead of at one from metal to air. The quantities r' and s' will therefore be altered, the new values being given, as before, by Fresnel's formulæ. This method of attacking the problem is adopted later (p. 313), when we are dealing with the more general case of incidence at any angle. As, however, we have already calculated the amplitude and phase of the reflected and transmitted beam for a metal plate surrounded by air, it is convenient to utilise these results in the present problem. To enable us to do this we can suppose the metal and glass separated by an *indefinitely thin* layer of air. An incident wave of unit amplitude gives rise to a reflected wave $R_1e^{i\theta_1}$, where R' and θ_1 are the R' and θ' already calculated for the reflected wave. It also produces a transmitted wave $R_2e^{i\theta_2}$ (R_2 and θ_2 having also been calculated). This wave is *immediately* reflected by the glass, giving a wave



$\sigma R_2e^{i\theta_2}$. $\sigma R_2{}^{i\theta_2}$ in its turn is transmitted through the metal plate and emerges into the air as a wave $(\sigma R_2e^{i\theta_2})\,R_2e^{i\theta_2} = Ae^{i\chi}$ say. The complete reflected wave is thus the resultant of the two components $R_1e^{i\theta_1}$ and $Ae^{i\chi}$. If, as before, we denote it by $R'e^{\theta'}$, we have

$$R'^2 = R_1{}^2+A^2+2R_1A\cos(\theta_1-\chi)$$

and

$$\tan\theta' = \frac{R_1\sin\theta_1+A\sin\chi}{R_1\cos\theta_1+A\cos\chi}.$$

If the incidence be direct, then σ is given by Fresnel's formula $\sigma = \dfrac{\mu-1}{\mu+1}$, where μ is the refractive index of the glass. Thus $A = \dfrac{\mu-1}{\mu+1}\cdot R_2{}^2$ and $\chi = 2\theta_2$.

For small thicknesses R_1 is very small, while R_2 is considerable. Under these circumstances R' is nearly equal to A and θ' to χ. As the thickness increases, R_2 rapidly diminishes, so that R' and θ' approximate more and more closely to R_1 and θ_1. Thus, except for very thin plates, the results will be the same as if the metal were surrounded by air instead of being deposited on glass. With very thin plates, however, the two cases will be quite different.

Taking $\mu=1{\cdot}54$ we get the following values of R'^2.

e_1/λ.	Indef. small.	0 ·001.	0 ·005.	0 ·01.	0 ·02.	0 ·05.	0 ·1.	0 ·2.	∞ .
Silver R'^2...........	0 ·0452	0 ·0419	0 ·1068	0 ·2411	0 ·5012	0 ·8525	0 ·9408	0 ·9516	0 ·9517
Gold R'^2	0 ·0452	0 ·0405	0 ·0621	0 ·1336	0 ·3327	0. 7025	0 ·8528	0 ·8794	0 ·8796

These results are represented graphically in fig. 1 above. Curve 4 corresponds to the silver and Curve 5 to the gold. A comparison of Curves 1 and 4 or Curves 2 and 5 will show the influence of the glass on the intensity of the reflected light.

The difference of phase (θ') is given in the following table.

e_1/λ.	Indef. small.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.	∞ .
Silver θ'	0	0·0946	0·3900	0·5356	0·6795	0·8005	0·8283	0·8312	0·8312
Gold θ'...............	0	0·0746	0·2886	0·4968	0·6427	0·7718	0·8056	0·8095	0·8095

Fig. 2 above gives a graphical representation of these results, Curve 4 for the silver and Curve 5 for the gold. As is to be expected, the difference between metal alone and metal on glass is very marked for small thicknesses. In all cases the change of phase is less for metal on glass than for metal alone, but the difference between the two cases diminishes rapidly as the thickness increases. After a thickness $e_1/\lambda = 0·05$ has been reached there is scarcely any appreciable difference between the results, and for the sake of clearness the curves for metal on glass have not been drawn beyond this point.

If we compare these results with Wiener's observations* of the change of phase produced by reflection from thin films of silver deposited chemically on glass, we see that there is a close agreement. The change of phase takes place rapidly, not so rapidly as with Wiener's first mirror (A), but more rapidly than with his second mirror (B). With the silver, discussed above, about 65 per cent. of the total change of phase is reached when $e_1/\lambda = 0·01$, *i.e.*, when $e_1 = 5·89 \times 10^{-6}$ mm. $= 5·89\ \mu\mu$. With Wiener's first mirror almost the whole change took place within a thickness $4\ \mu\mu$, but with the second mirror the corresponding thickness was $12\ \mu\mu$.

So far we have been occupied mainly with a discussion of the intensity and phase of the reflected and transmitted† light in the case of direct incidence. We shall now return to the more general problem stated at the onset, that of determining the ellipticity and the difference of phase between the parallel and perpendicularly polarised light for any angle of incidence. In the reflected light we have seen (p. 298) that $R'e^{i\theta'} = r(1-q^2)/(1-q^2r^2)$.

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* 'Wied. Ann.,' vol. 31, p. 629.

† We have said nothing of the influence of the glass on the transmitted beam. It is shown later (p. 337) that for direct incidence the ellipticity is not affected by the glass.

Hence

$$\frac{R_1'}{R_2'}e^{i(\theta_1'-\theta_2')} = \frac{r_1(1-q^2r_2^2)}{r_2(1-q^2r_1^2)} = \frac{R_1}{R_2}e^{i(\theta_1-\theta_2)}\frac{1-QR_2^2e^{i(2\theta_2-\psi)}}{1-QR_1^2e^{i(2\theta_1-\psi)}}.$$

Thus if ϵ be the ellipticity and ω the azimuth of the reflected light we have

$$\tan^2\omega = \epsilon^2 = \frac{R_1^2}{R_2^2}\,\frac{1+Q^2R_2^4-2QR_2^2\cos(\psi+\gamma_2)}{1+Q^2R_1^2-2QR_1^2\cos(\psi+\gamma_1)}.$$

As the thickness of the plate increases, Q diminishes and ϵ approaches the value R_1/R_2. We shall calculate ϵ for different thicknesses of the gold and silver already discussed.

If Δ be the difference of phase, we have $\Delta = \theta_2-\theta_1+\Phi_1-\Phi_2$, where Φ is given by the formula

$$\tan\Phi = \frac{QR^2\sin(\psi+\gamma)}{1-QR^2\cos(\psi+\gamma)}.$$

Having obtained Δ for a series of angles, we can represent the results graphically and so determine the angle of incidence that makes Δ equal to a quarter of the wave-length. This is the *Principal Incidence* and the corresponding value of ω is the *Principal Azimuth.*

We proceed to tabulate the values of the constants occurring in these formulæ for the silver whose optical constants are $\mu_1 = 0{\cdot}18$ and $a = 3{\cdot}67$. If $\mu = \mu_1 - ia = Me^{-i\alpha}$, we have $M^2 = 13{\cdot}5$ and $\alpha = 87^\circ\ 12'$. The other quantities are to be calculated from the formulæ

$$\sin\phi = \mu\sin\phi'; \qquad \cos\phi' = ce^{-iu}; \qquad \tan 2u = \frac{M^{-2}\sin^2\phi\sin 2\alpha}{1-M^{-2}\sin^2\phi\cos 2\alpha};$$

$$c^4 = 1-2M^{-2}\sin^2\phi\cos 2\alpha + M^{-4}\sin^4\phi;$$

$$m^4 = M^4+\sin^4\phi-2M^2\cos 2\alpha\sin^2\phi; \qquad \tan 2\chi = \frac{M^2\sin 2\alpha}{M^2\cos 2\alpha-\sin^2\phi}.$$

The formulæ for Q, ψ, θ_1, θ_2, R_1, and R_2 have been already given on p. 298.

From these we get the following table. The values of log Q and ψ in this and later tables refer to the case $e_1/\lambda = 0{\cdot}1$. As these quantities are directly

ϕ.	20°.	40°.	60°.	65°.	70°.	75°.	80°.	85°.
u	1′	6′	10′	11′	11′	12′	12′	12′
c	1·004	1·015	1·028	1·030	1·032	1·034	1·035	1·035
$\frac{1}{2}\gamma_1 = 180^\circ-\theta_1$	32° 25′	39° 38′	58° 23′	67° 4′	78° 46′	94° 48′	116° 45′	145° 40′
$\frac{1}{2}\gamma_2 = 180^\circ-\theta_2$	28° 33′	23° 12′	15° 5′	12° 44′	10° 18′	7° 47′	5° 14′	2° 37′
R_1^2	0·9491	0·9378	0·9156	0·9084	0·9028	0·9010	0·9105	0·9423
R_2^2	0·9548	0·9636	0·9766	0·9802	0·9842	0·9879	0·9919	0·9961
χ	87° 14′	87° 16′	87° 21′	87° 21′	87° 22′	87° 23′	87° 24′	87° 24′
m	3·691	3·730	3·775	3·784	3·792	3·798	3·804	3·806
ψ	12° 50′	12° 50′	12° 32′	12° 32′	12° 32′	12° 30′	12° 30′	12° 30′
log Q	$\bar{3}$·989	$\bar{3}$·967	$\bar{3}$·942	$\bar{3}$·936	$\bar{3}$·933	$\bar{3}$·929	$\bar{3}$·926	$\bar{3}$·925

proportional to the thicknesses, their values in other cases can be readily obtained by simple division or multiplication.

From these numbers and the formulæ above we derive the following table, giving the value of Δ expressed as a fraction of the half wave-length :—

e_1/λ.	$\phi = 20°$.	40°.	60°.	70°.	75°.	80°.	85°.
0·001	0·0404	0·1723	0·4524	0·7039	0·8966	1·132	1·404
0·005	0·0334	0·1428	0·3723	0·5821	0·7333	0·9245	1·148
0·01	0·0269	0·1150	0·3056	0·4922	0·6333	0·8222	1·045
0·02	0·0207	0·0900	0·2531	0·4269	0·5588	0·7365	0·9389
0·05	0·0204	0·0874	0·2373	0·3878	0·4958	0·6506	0·8330
0·1	0·0214	0·0909	0·2404	0·3814	0·4848	0·6227	0·7977
0·2	0·0214	0·0913	0·2405	0·3804	0·4834	0·6193	0·7943

These results are exhibited graphically in figs. 5 and 6 below.

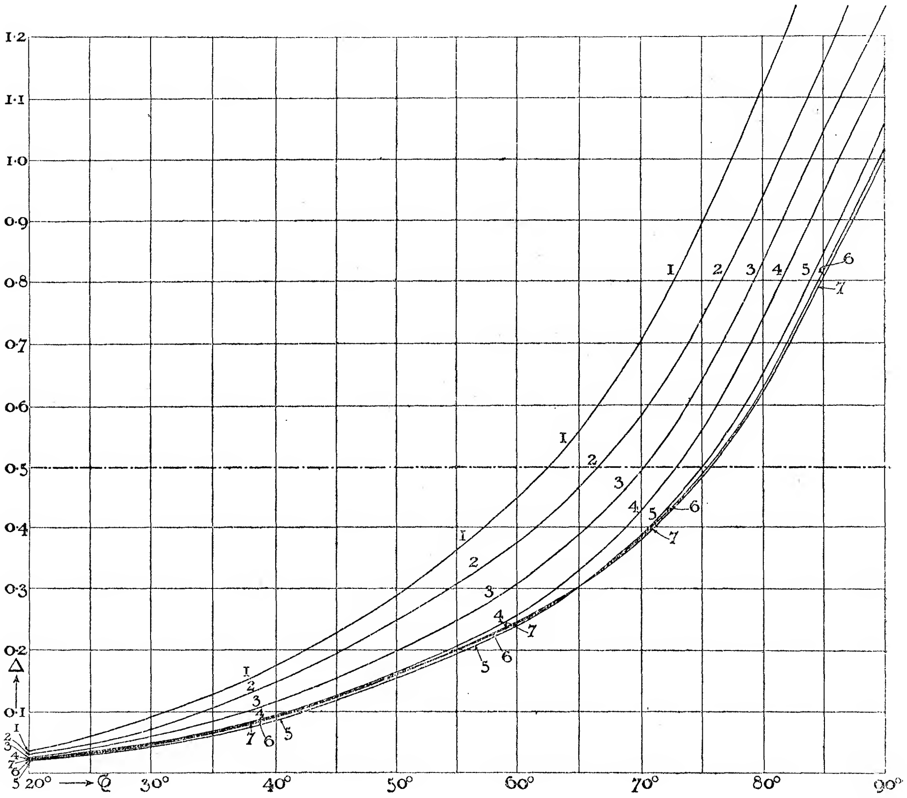


FIG. 5.

Fig. 5 represents Δ as a function of ϕ, the angle of incidence. The curves numbered 1 to 7 correspond to the seven different values of e_1/λ

in the above table, from 0·001 to 0·2. It will be seen that in all cases the difference of phase increases with the angle of incidence. From the graph we see that the values of the Principal Incidence for the various thicknesses dealt with are 62° 40′, 66° 30′, 70° 15′, 73°, 74° 30′, 75° 30′, 75° 48′. The Principal Incidence increases with the thickness, rapidly at first, but afterwards more slowly as it tends to the constant value appropriate to a thick plate. There is little further change in the value of the Principal Incidence when the thickness of the metal exceeds 0·00003 mm.

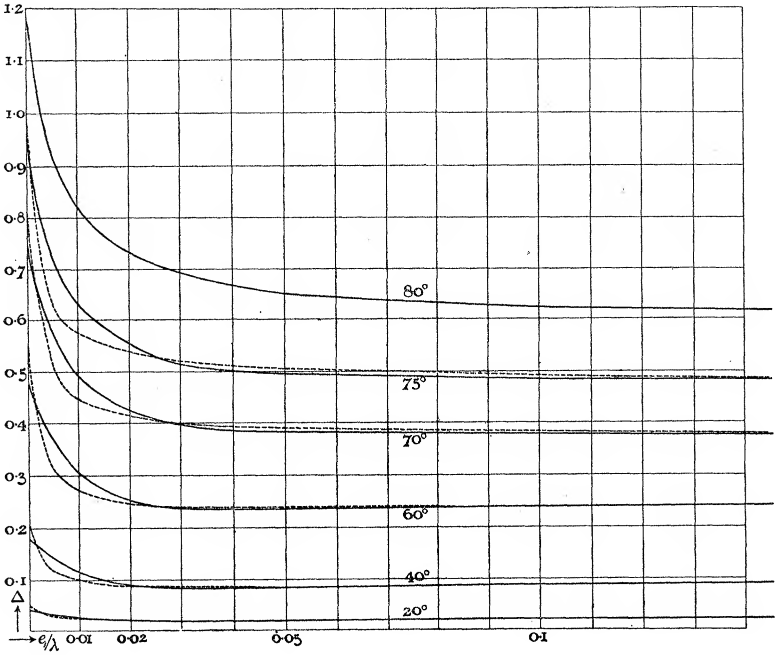


Fig. 6.

The continuous curves in fig. 6 above represent Δ as a function of the thickness for different angles of incidence. (The dotted curves refer to the case of silver films on glass, to be dealt with later.) These curves show how rapidly Δ diminishes as the thickness increases at first. For incidences of 60° and under, Δ begins by diminishing, and then very slowly increases again. The minimum value of Δ occurs when e_1/λ is in the neighbourhood of 0·05. For incidences of 70° and upwards there is no such minimum, and Δ continually diminishes until it reaches the value appropriate to a thick plate.

The azimuth (ω) is given in the following table :—

e_1/λ.	$\phi = 20°$.		40°.		60°.		70°.		75°.		80°.		85°.	
	°	′	°	′	°	′	°	′	°	′	°	′	°	′
0·001	41	42	31	47	17	9	10	29	7	58	6	7	7	38
0·005	41	57	32	35	18	48	12	40	10	36	9	49	12	50
0·01	42	22	34	19	22	15	16	57	15	28	15	52	22	0
0·02	43	18	38	0	29	35	25	41	24	48	26	16	33	30
0·05	44	39	43	7	40	34	39	8	39	0	39	57	42	37
0·1	44	53	44	29	43	46	43	21	43	15	43	28	44	7
0·2	44	55	44	37	44	5	43	46	43	41	43	46	44	12

These results are represented in figs. 7 and 8 below.

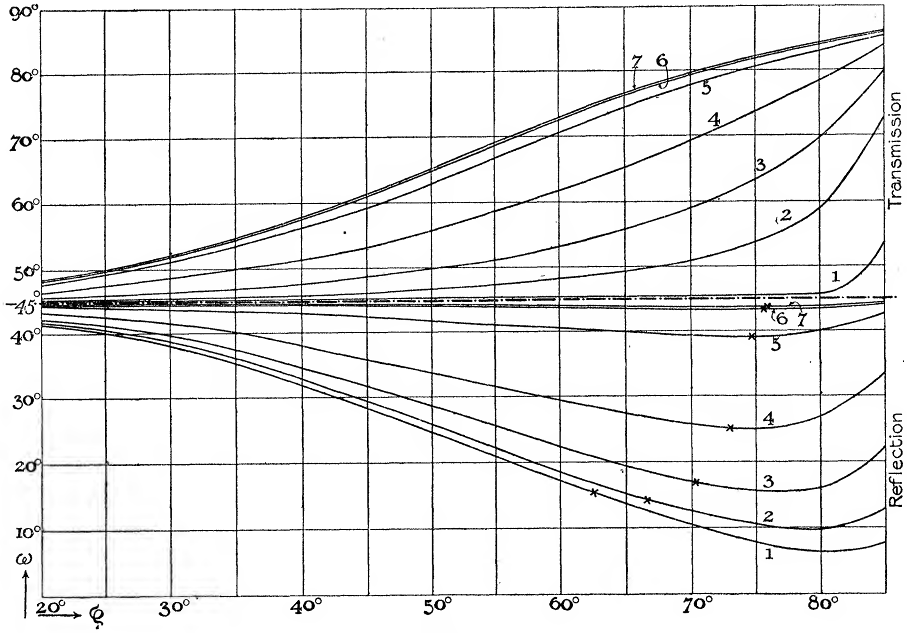


FIG. 7.

It will be seen that the azimuth is always less than 45°, so that for reflection R_1' is always less than R_2'. The case of reflection is represented by the part of the figure below the line $\omega = 45°$. Fig. 7 depicts ω as a function of the incidence for different thicknesses. It shows that the azimuth diminishes as the incidence increases, reaches a minimum, and then increases. The incidence at which ω is a minimum decreases as the thickness of the metal increases. From the figure we easily obtain the values of the Principal Azimuth for the different thicknesses, using the results already obtained for the Principal Incidence. The Principal Azimuth proves to be

15° 20′, 14°, 16° 45′, 24° 50′, 39° 10′, 43° 20′, and 43° 50′ in the different cases. It thus diminishes at first as the thickness increases, then increases rapidly until a thickness of 1/10 of the wave-length is reached, after which there is very little increase as the plate thickens. The values of the Principal Incidences and Principal Azimuths are indicated by the positions of the crosses (×) in the above figure.

Fig. 8 below represents the same quantity ω as a function of the thickness of the plate for different incidences. It shows how the azimuth increases with the thickness for values of e_1 less than 1/10 of the wave-length, and that beyond this there is very little change in ω.

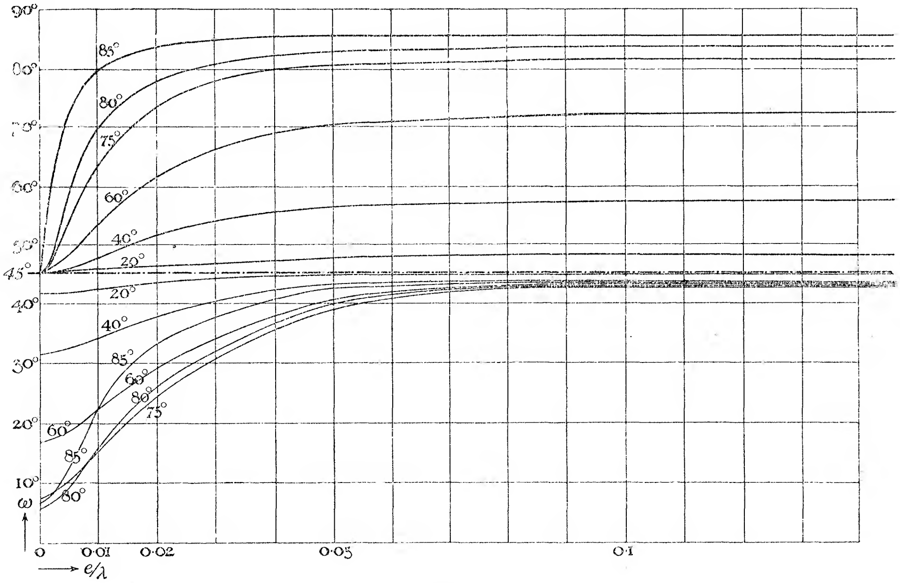
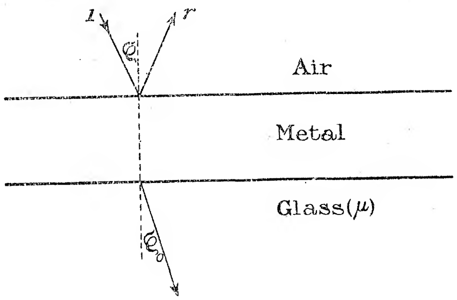


FIG. 8.

Before attempting to compare these results with experiment, we must see to what extent they are modified when the silver is deposited on glass, for it is under such conditions that most of the careful observations have been made.



In this case the reflected wave, instead of being $r(1-q^2)/(1-q^2r^2)$, is now $(r-q^2\rho)/(1-q^2r\rho)$, where r is the same as before, and ρ the corresponding quantity, when a ray of light goes from glass to metal instead of from air to metal. The quantity q is the same as before, and in calculating ρ we must replace M by M/μ and ϕ by ϕ_0, where $\sin\phi = \mu\sin\phi_0$ and μ is the coefficient of refraction of the glass. Putting $\rho = Pe^{i\omega}$ and taking $\mu = 1{\cdot}54$, we get the following set of values:—

ϕ.	20°.	40°.	60°.	65°.	70°.	75°.
$180°-\omega_1$	48° 13′	58° 3′	81° 27′	91° 13′	103° 19′	118° 20′
$180°-\omega_2$	42° 48′	35° 5′	23° 2′	19° 30′	15° 49′	11° 58′
P_1^2	0·9290	0·9177	0·9028	0·9010	0·9033	0·9118
P_2^2	0·9363	0·9475	0·9649	0·9705	0·9750	0·9833

We have
$$\frac{R_1'}{R_2'}e^{i(\theta_1'-\theta_2')} = \frac{r_1-q^2\rho_1}{r_2-q^2\rho_2}\cdot\frac{1-q^2r_2\rho_2}{1-q^2r_1\rho_1}.$$

Thus the ellipticity (ϵ) is given by the formula

$$\epsilon^2 = \frac{R_1^2+Q^2P_1^2-2QR_1P_1\cos(\omega_1-\theta_1-\psi)}{R_2^2+Q^2P_2^2-2QR_2P_2\cos(\omega_2-\theta_2-\psi)} \times\frac{1+Q^2R_2^2P_2^2-2QR_2P_2\cos(\theta_2+\omega_2-\psi)}{1+Q^2R_1^2P_1^2-2QR_1P_1\cos(\theta_1+\omega_1-\psi)},$$

and the difference of phase (Δ) is

$$\chi_2-\chi_1+\Phi_1-\Phi_2,$$

where

$$\tan\chi = \frac{R\sin\theta-QP\sin(\omega-\psi)}{R\cos\theta-QP\cos(\omega-\psi)} \quad\text{and}\quad \tan\Phi = \frac{-QRP\sin(\theta+\omega-\psi)}{1-QRP\cos(\theta+\omega-\psi)}.$$

From these formulæ and the constants above we derive the following table for Δ.

e_1/λ.	ϕ = 20°.	40°.	60°.	65°.	70°.	75°.
0·001	0·0435	0·1833	0·4657	0·5732	0·6988	0·8567
0·005	0·0297	0·1186	0·3131	0·3967	0·4957	0·6294
0·01	0·0254	0·1014	0·2733	0·3479	0·4446	0·5744
0·02	0·0207	0·0901	0·2491	0·3204	0·4156	0·5419
0·05	0·0200	0·0884	0·2401	0·3057	0·3922	0·5086
0·1	0·0213	0·0911	0·2404	0·3024	0·3819	0·4856
0·2	0·0214	0·0913	0·2405	0·3018	0·3804	0·4834

These results are represented in fig. 6 above and fig. 9 below. The dotted curves in fig. 6 exhibit Δ as a function of the thickness of the film for different angles of incidence. The continuous curves of that figure represent the corresponding values of Δ when there is no glass, so that a comparison of the two curves shows at once the modifications introduced by depositing the metal on glass. Quincke concluded from his experiments* that the difference of phase is always less for silver alone than for the same thickness of silver on glass. This agrees with our results for thicknesses greater than that given by $e_1/\lambda = 0{\cdot}03$, and it was on films of such thickness that Quincke experimented. The excess of Δ for silver on glass over its value for silver alone increases with the angle of incidence, but is always small. We see, moreover, that Quincke's conclusion is not applicable to films whose thickness is less than 0·02 λ, except for *excessively* thin films.

Fig. 9 represents Δ as a function of ϕ for different thicknesses. The curve marked 0 corresponds to an *indefinitely* thin film of silver on the glass, the other curves to the different thicknesses mentioned in the table above. From the graph we obtain the following values of the Principal Incidences:—57°, 61° 35′, 70° 12′, 72°, 73° 25′, 74° 45′, 75° 40′, 75° 48′. The angle 57° is

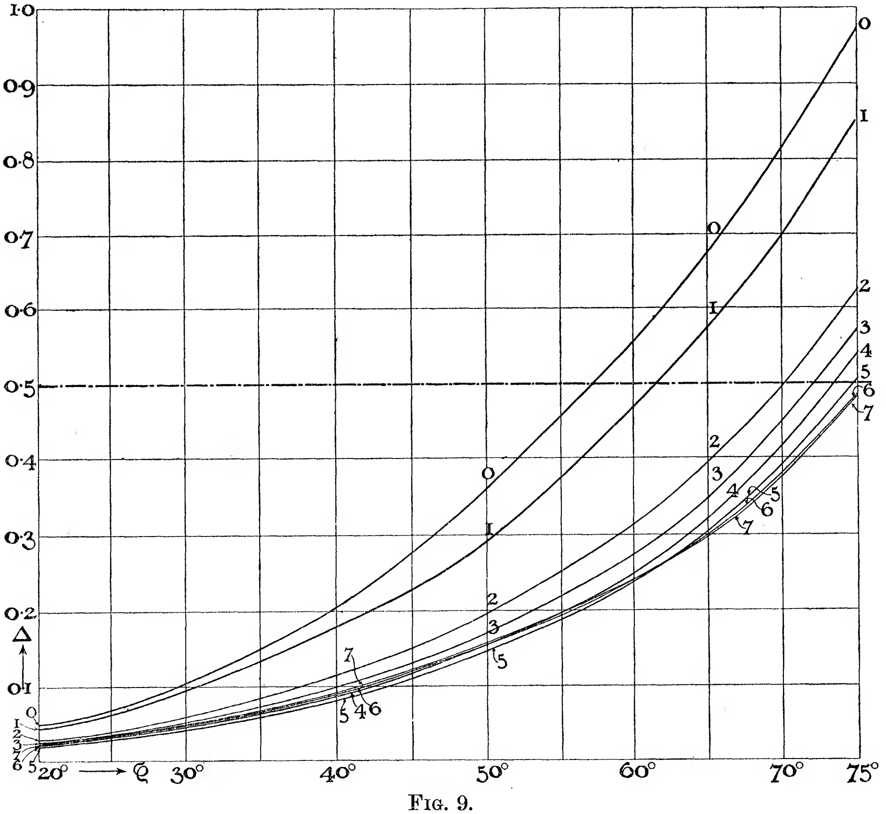


Fig. 9.

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* 'Pogg. Ann.,' vol. 119, p. 368.

the polarising angle of the glass, and we thus see that we can pass from vitreous reflection to metallic reflection through all the intermediate states by the aid of layers of silver gradually increasing in thickness. One of the most striking facts about these results is the very rapid change in the Principal Incidence produced by a slight thickening of the silver film when the film is very thin. A film of only one-thousandth of the wave-length increases the Principal Incidence from 57° to 61° 35′. This indicates how considerable would be the influence of a surface layer of any kind on the position of the Principal Incidence and serves to explain the observation of all careful experimenters that the Principal Incidence alters with different conditions of the reflecting surface.* It is also in good agreement with an experiment by Mascart† on the subject. A silver film was deposited on glass by electrolysis. The film was so thin that its existence could not have been suspected without previous knowledge. It was quite impossible to estimate the thickness exactly, but it was certainly not greater than five-thousandths of a wave-length. Even with this thin film the properties of the light reflected from the glass were considerably modified and the Principal Incidence had increased from 57° to 59° 45′. Conroy made various experiments‡ with silver films on glass. The films varied in thickness from $4{\cdot}035\times10^{-6}$ cm. to $8{\cdot}995\times10^{-6}$ cm. Taking $\lambda = 5{\cdot}89\times10^{-5}$ cm. for yellow light, this makes e_1/λ vary from 0·069 to 0·153. This range lies between Curves 5 and 7 of the figure above. Our graph makes the Principal Incidence increase in this range from 74° 45′ to 75° 48°, an increase of 1° 3′. Conroy found an increase of 2° 4′. Although the Principal Incidence changes rapidly at first, it soon tends to a constant value and there is little further change when the thickness exceeds 0·00003 mm. Quincke§ made a large number of observations with silver films and red light. He found that the values of the Principal Incidence increased with the thickness, tended to a constant value and changed very little when the thickness of the silver film exceeded 0·00002 mm. To exhibit more clearly the influence of the glass on the Principal Incidence the results for silver alone and for silver on glass are set out together and their differences stated.

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* *Cf.* 'Roy. Soc. Proc.,' 1906, p. 211.

† 'Comptes Rendus,' vol. 76, p. 866.

‡ 'Roy. Soc. Proc.,' 1881, p. 486.

§ 'Pogg. Ann.,' vol. 129, p. 177.

e_1/λ.	0·001.		0·005.		0·01.		0·02.		0·05.		0·1.		0·2.	
	°	′	°	′	°	′	°	′	°	′	°	′	°	′
P.I. (silver alone)	62	40	66	30	70	15	73	0	74	30	75	30	75	48
P.I. (silver on glass)	61	35	70	12	72	0	73	25	74	45	75	40	75	48
Difference	−1	5	+3	42	+1	45	+0	25	+0	15	+0	10	0	

It appears from this that, except for excessively thin films, the Principal Incidence is greater for silver on glass than for the same thickness of silver alone. The difference, however, is nowhere very considerable, and it diminishes to zero as the thickness of the metal increases.

The ellipticity (ϵ) is given by the formula on p. 313; it is most conveniently expressed as the tangent of the azimuth, whose value is set out in the following table:—

e_1/λ.	ϕ = 20°.		40°.		60°.		65°.		70°.		75°.	
	°		°	′	°	′	°	′	°	′	°	′
0·001	44	47	44	25	43	20	42	37	41	29	40	13
0·005	43	42	39	51	32	6	29	5	26	3	22	46
0·01	43	28	37	37	28	22	25	50	22	40	19	54
0·02	43	21	38	38	31	19	29	34	28	15	26	41
0·05	44	29	42	59	40	28	40	1	40	9	41	22
0·1	44	51	44	27	43	44	43	22	43	23	43	24
0·2	44	54	44	36	44	5	43	23	43	46	43	41

Fig. 10 below exhibits these results graphically.

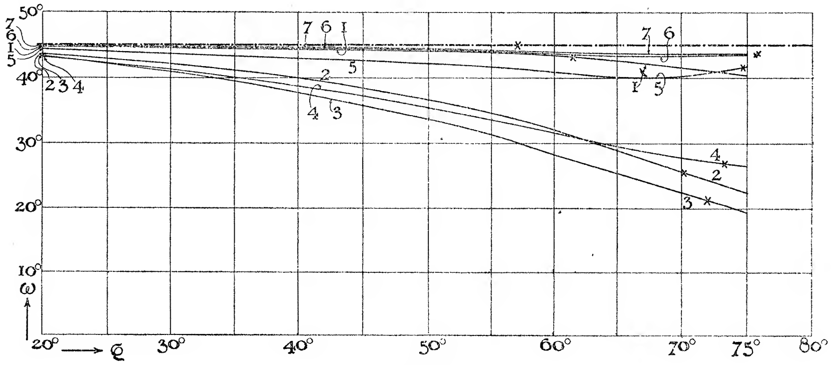

FIG. 10.

A comparison of figs. 7 and 10 will indicate clearly the change in the azimuth produced by depositing the silver on glass. It will be seen that Curves 4, 5, 6, and 7 in the two figures are very similar, so that after a thickness 0·02 λ has been reached it does not make a great difference whether the silver is alone or resting on glass. As is to be expected, however, with very thin films the difference between the two cases is marked. On comparing Curves 1, 2, and 3 in the two figures we see that they are completely different. For silver alone the azimuth *increases* with the thickness, whereas for silver films on glass it *diminishes* rapidly at first. The Principal Incidence having been obtained above it is easy to determine the corresponding Principal Azimuth from the figure. The position of the Principal Incidence and Azimuth is marked in fig. 10 by a cross (×). The following table gives the Principal Azimuth for different thicknesses and compares it with the case of silver alone:—

e_1/λ.	0·001.		0·005.		0·01.		0·02.		0·05.		0·1.		0·2.	
	°	′	°	′	°	′	°	′	°	′	°	′	°	′
P.A. (silver alone)…………	15	20	14	0	16	45	24	50	39	10	43	20	43	50
P.A. (silver on glass) ……	43	15	25	40	21	20	27	10	41	15	43	30	43	50
Difference ……………	27	55	11	40	4	35	2	20	2	5	0	10	0	

The Principal Azimuth is thus always greater for silver on glass than for silver alone. The difference is very considerable for small thicknesses, but diminishes rapidly to zero as the thickness increases. When the thickness changes from 0·05 λ to 0·2 λ the Principal Azimuth increases 2° 25′. Conroy* found a much larger increase, viz., 14°, in the same range. It appears, however, that the variation of the Principal Azimuth with the thickness is very considerable in the neighbourhood of $e_1/\lambda = 0{\cdot}05$. When the thickness increases from 0·02 λ to 0·05 λ there is a change of 14° 5′ in the Principal Azimuth. Thus a slight error in the estimate of the thickness of the thinnest of Conroy's films would account for the apparent discrepancy between theory and experiment.†

Having discussed the nature of the light reflected from silver films, we proceed to estimate the extent to which these results are modified when films of some other metal are employed. So far our results have been obtained on the supposition that the transition from air to the metal is abrupt and not gradual. This, of course, is only an approximation to the truth; but in the case of silver the approximation has proved to be sufficiently close for colligating the experimental results. When the influence of the layer of transition is considered it is found* that the difference of phase is altered by a quantity χ which is a maximum near the Principal Incidence, where it is given by the formula

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* Conroy, *loc. cit.*, p. 315.

† The thickness was calculated from the weight of a measured area. There was necessarily some uncertainty as to the density of the silver in so finely divided a state.

$$\tan\chi = \frac{\sin^4\phi - M^2c^2\cos^2\phi}{2Mc\cos\phi\sin^2\phi\sin(\alpha+u)}.$$

This enables us to determine for what metals the correction due to the layer of transition is most considerable; *e.g.*, it would lead us to expect that the influence of the layer would be greater with gold than with silver. As, however, the analysis is necessarily much more complex when there is a surface layer to be taken into account, we shall hold fast to the hypothesis of a sudden transition from one medium to the other as a sufficiently close approximation to the truth, unless we are driven from it by too great a discrepancy between theory and observation.

The metal that lends itself most readily to accurate experiments with thin plates is gold. Working with red light, Conroy found a Principal Incidence of 74° and a Principal Azimuth of 42° as the mean of several determinations. On the supposition of a sudden transition from air to gold the optical constants can be derived from these angles by means of the formulæ†

$$Mc = \frac{\sin^2\phi}{\cos^2\phi}, \qquad \omega = \frac{\alpha+u}{2},$$

coupled with the formulæ for c and u already given (p. 308). In this way we obtain $M^2 = 10{\cdot}4$ and $\alpha = 83°\ 30'$. With these constants and the formulæ on p. 308 we get the following table:—

ϕ.	30°.	40°.	50°.	60°.	70°.	80°.	85°.
u	9′	15′	21′	26′	30′	33′	34′
c	1·012	1·020	1·027	1·035	1·041	1·045	1·046
$180-\theta_1$	27° 36′	31° 55′	39° 26′	53° 47′	84° 38′	136° 41′	160° 2′
$180-\theta_2$	19° 43′	17° 12′	14° 12′	10° 53′	7° 22′	3° 43′	1° 52′
R_1^2	0·9105	0·9007	0·8867	0·8708	0·8570	0·8782	0·9251
R_2^2	0·9311	0·9414	0·9510	0·9614	0·9743	0·9865	0·9940
χ	83° 39′	83° 45′	83° 51′	83° 56′	84°	84° 3′	84° 4′
m	3·264	3·288	3·312	3·336	3·357	3·369	3·372
ψ	26°	25° 46′	25° 34′	25° 24′	25° 15′	25° 8′	25° 4′
log Q	$\bar{2}$·230	$\bar{2}$·217	$\bar{2}$·203	$\bar{2}$·190	$\bar{2}$·178	$\bar{2}$·172	$\bar{2}$·170

With the aid of these quantities we derive the following series of values of Δ from the formulæ of p. 308:—

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* See 'Roy. Soc. Proc.,' 1906, p. 224.

† *Loc. cit.*, p. 213.

e_1/λ.	ϕ = 30°.	40°.	50°.	60°.	70°.	80°.
0·001	0·0790	0·1490	0·2550	0·4323	0·7793	1·323
0·005	0·0605	0·1147	0·1977	0·3413	0·6422	1·110
0·01	0·0457	0·0852	0·1548	0·2797	0·5577	1·009
0·02	0·0363	0·0698	0·1261	0·2372	0·4945	0·9093
0·05	0·0403	0·0758	0·1324	0·2333	0·4498	0·7854
0·1	0·0435	0·0777	0·1398	0·2389	0·4331	0·7444
0·2	0·0439	0·0818	0·1403	0·2391	0·4292	0·7389

These results are represented in fig. 11 below.

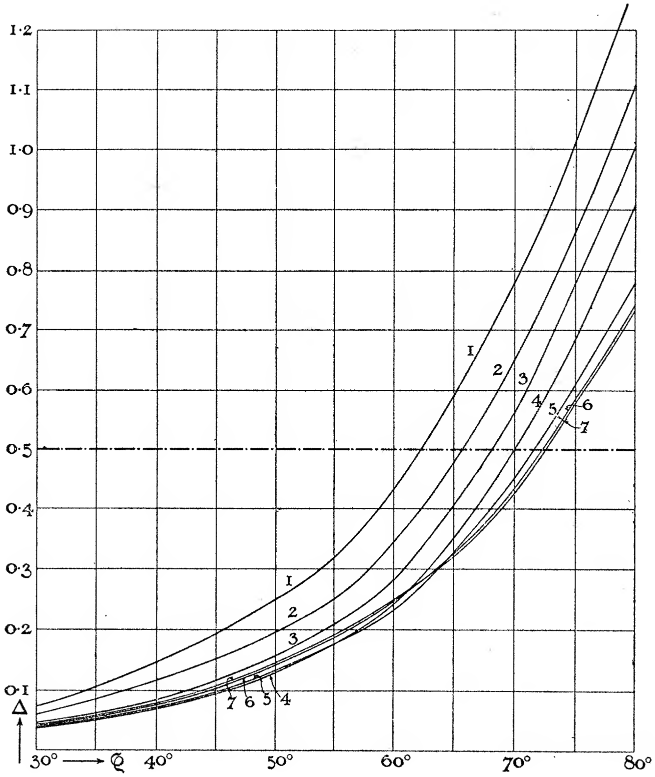

Fig. 11.

A comparison with fig. 5 will show that there is not much difference between the two metals as far as the difference of phase produced by reflection is concerned. Δ follows very much the same law in the two cases, as regards its dependence both on the angle of incidence and the thickness of the plate. The Principal Incidences corresponding to the various thicknesses are 62° 24′, 65° 40′, 68° 12′, 70°, 71° 45′, 72° 20′, and 72° 45′. The main difference is that the Principal Incidence is rather slower to reach its final value for gold than for silver. For silver we found that when $e_1/\lambda=0{\cdot}2$ the Principal Incidence was the same as for a thick plate; for gold, however, when $e_1/\lambda=0{\cdot}2$ the Principal Incidence is still 1° 15′ from its final value.* Conroy experimented with gold leaves varying in thickness from $6{\cdot}45\times10^{-5}$ cm. to $1{\cdot}135\times10^{-5}$ cm. For red light, taking $\lambda=7{\cdot}6\times10^{-5}$ cm., this makes e_1/λ vary from 0·085 to 0·149. This range lies between the Curves 5 and 7 of fig. 11, and we see that the variation of the Principal Incidence is small. The increase is only about a degree, whereas Conroy found an increase of 1° 47′. The agreement is quite as close as could be expected considering the discrepancies between the different experimental results, the uncertainty as to the optical constants and the specific gravity† of the metal in the state of gold leaf, and the fact that we have neglected the influence of the glass on which the leaves were placed.‡ Conroy's conclusion that the Principal Incidence always increases with the thickness of the metal is supported by our theory with the optical constants that we have adopted for silver and gold. However, Meslin's experiments with gold over a much wider range of thickness show that the law is not universally applicable, and we shall see later that this is supported by theory when the transition layer is taken into consideration.

The azimuth (ω) is given in the following table:—

e_1/λ.	ϕ = 30°.		40°.		50°.		60°.		70°.		80°.		85°.	
	°	′	°	′	°	′	°	′	°	′	°	′	°	′
0·001	36	12	29	26	21	24	13	29	7	26	6	0	6	59
0·005	37	3	30	49	23	21	15	48	10	13	10	57	17	55
0·01	38	24	33	9	26	42	19	48	14	34	18	0	28	32
0·02	40	49	37	23	33	2	29	0	23	2	28	48	38	14
0·05	43	44	42	38	41	8	39	5	37	15	39	5	43	25
0·1	44	33	44	10	43	38	43	3	41	40	42	40	43	58
0·2	44	41	44	22	44	0	43	35	42	10	43	0	43	58

These results are exhibited in fig. 12 below.

On comparing this with fig. 7 above it will be seen that there is not much difference between the behaviour of the two metals. The Principal Azimuths for the various thicknesses are found from the figure to be 11° 45′, 12°, 15° 27°, 37° 20′, 41° 40′, and 42° respectively. The crosses (×) in the figure represent the positions of the Principal Incidence and Azimuth. From the

---

* It must be remembered that λ is not the same in the two cases, the incident light being yellow for silver and red for gold.

† This, of course, affects the estimate of the thickness, and a slight change in this would easily account for the apparent difference between theory and experiment.

‡ *Cf.* p. 316. For such thicknesses the effect of the glass is small.

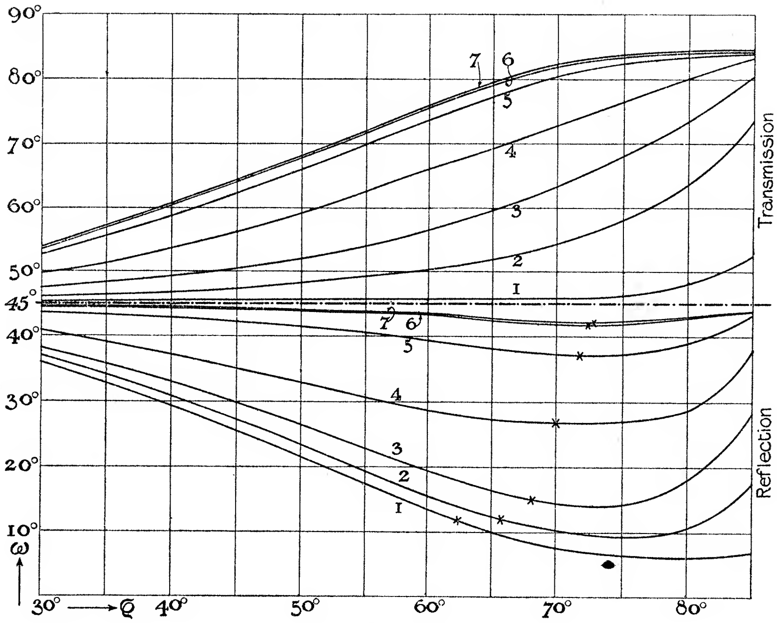

FIG. 12.

results of our investigation in the case of silver we should expect these angles to be very considerably altered if the gold were deposited on glass. The difference, however, would not be great for the thicknesses greater than 0·05 λ. It was with such thicknesses that the experiments of Conroy, referred to above, were carried out. Between the Curves 5 and 7 of the above figure the Principal Azimuth increases 2° 40′. Within the same range Conroy found an increase of about 2°. Conroy concluded that the Principal Azimuth always increases with the thickness. Our theory leads to a similar conclusion for a thin plate of gold surrounded by air, but if the gold were placed on glass the law of variation of the Principal Azimuth would be completely changed, as was found in the case of silver. The theory, however, is in satisfactory agreement with Conroy for thicknesses such as he employed in his experiments.

Transmission.

We shall now turn from the reflected light to consider in a similar manner the nature of the transmitted beam. We have seen (p. 298) that the transmitted beam is represented by $q(1-r^2)/(1-q^2r^2)$. Thus, with the same notation as in the case of reflection, we have

$$\frac{R_1'}{R_2'}e^{i(\theta_1'-\theta_2')} = \frac{1-r_1^2}{1-r_2^2}\,\frac{1-q^2r_2^2}{1-q^2r_1^2}.$$

Proceeding as on pp. 307 and 308, we get

$$\tan^2\omega = \epsilon^2 = \frac{1+R_1^4-2R_1^2\cos\gamma_1}{1+R_2^4-2R_2^2\cos\gamma_2}\cdot\frac{1+Q^2R_2^4-2QR_2^2\cos(\psi+\gamma_2)}{1+Q^2R_1^2-2QR_1^2\cos(\psi+\gamma_1)},$$

and $$\Delta = \Phi_1-\Phi_2-(\Phi_1'-\Phi_2'),$$

where Φ is the same as on p. 308 and Φ' is obtained from Φ by putting $Q = 1$ and $\psi = 0$.

From these formulæ and the constants already set out we derive the following set of values for the difference of phase in the case of *Silver*:—

e_1/λ.	$\phi = 20°$.	40°.	60°.	75°.	80°.	85°.
0·001	0·0019	0·0084	0·0177	0·0546	0·0791	0·1423
0·005	0·0085	0·0386	0·0978	0·2179	0·2922	0·3930
0·01	0·0154	0·0664	0·1644	0·3167	0·3941	0·5009
0·02	0·0170	0·0915	0·2168	0·3834	0·4778	0·6181
0·05	0·0220	0·0939	0·2328	0·4328	0·5658	0·7201
0·1	0·0208	0·0904	0·2296	0·4667	0·5939	0·7484
0·2	0·0207	0·0900	0·2295	0·4679	0·5973	0·7513

These results are represented in figs. 13 and 14 below.

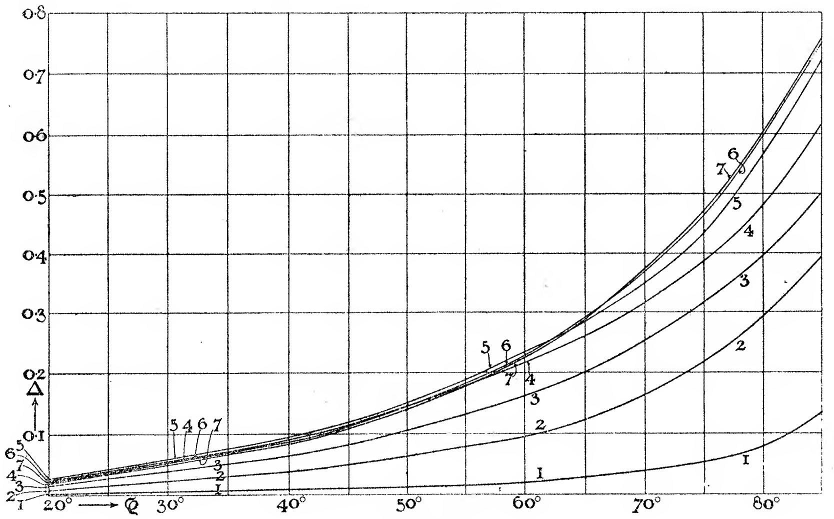

Fig. 13.

Fig. 13 shows the difference of phase (Δ) as a function of the angle of incidence (ϕ). The difference of phase increases with the incidence, at first slowly, but afterwards rapidly, especially when the thickness of the plate is not very small. The next figure (14) indicates more clearly in what way Δ depends on the thickness. For incidences up to 60°, Δ increases fairly rapidly at first, then more slowly until a maximum is reached, and then declines very slowly to the value appropriate to a thick plate. For larger incidences the increase of Δ is much more rapid at first, and the increase continues throughout, although as the thickness increases it becomes very slow, until it reaches the value corresponding to a thick plate. As no very careful measurements have been made for transmission through silver,* we cannot test these results by comparing them with experiment.

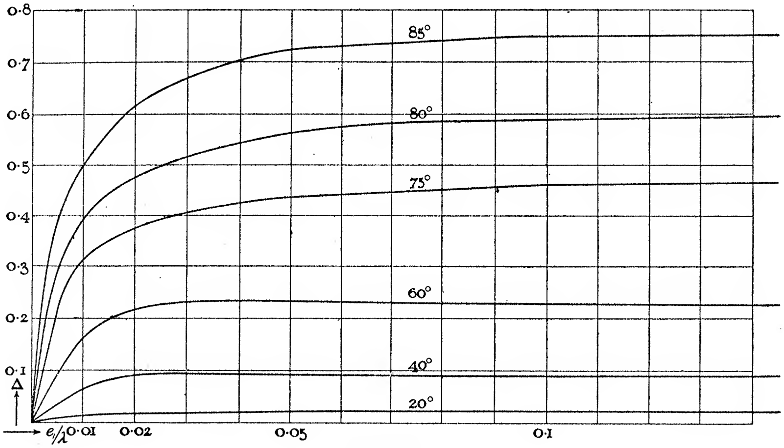

Fig. 14.

In the azimuth (ω) we get the following table:—

e_1/λ.	ϕ = 20°.		40°.		60°.		75°.		80°.		85°.	
	°	′	°	′	°	′	°	′	°	′	°	′
0·001	45	1	45	6	45	19	45	43	45	49	53	36
0·005	45	14	45	59	48	7	54	0	58	59	72	17
0·01	45	40	47	52	53	17	63	51	69	53	79	47
0·02	46	36	51	41	61	44	73	37	78	5	83	44
0·05	47	51	56	36	70	23	80	29	82	55	85	28
0·1	48	10	57	51	72	19	81	48	83	44	85	42
0·2	48	12	57	58	72	31	81	54	83	48	85	43

These results are represented in figs. 7 and 8 above. The azimuth is always greater than 45°. It increases slowly and almost uniformly as the incidence increases, except for large values of ϕ, when the increase is more rapid, especially for very thin films. The azimuth also increases with the thickness, very rapidly at first, especially for large values of ϕ; but after the thickness is greater than 0·05 λ the increase is very slow.

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[*] Meslin made a few such measurements, but did not record the thickness of the silver. His experiments with gold will come under discussion later.

Similar results are obtained for *gold* with the constants already set out. In this way we get the following table for the difference of phase (Δ).

e_1/λ.	ϕ = 30°.	40°.	50°.	60°.	70°.	80°.	85°.
0·001	0·0055	0·0239	0·0384	0·0482	0·0442	0·0409	0·0406
0·005	0·0101	0·0445	0·0712	0·0893	0·0833	0·0779	0·0776
0·01	0·0171	0·0743	0·1167	0·1460	0·1390	0·1311	0·1317
0·02	0·0280	0·1180	0·1805	0·2231	0·2262	0·2214	0·2218
0·05	0·0547	0·2300	0·3143	0·3874	0·4260	0·4361	0·4417
0·1	0·0992	0·3125	0·4140	0·5141	0·6371	0·6778	0·6837
0·2	0·1812	0·4381	0·5554	0·6775	0·7848	0·8112	0·8156

These results are set out graphically in fig. 15 below.

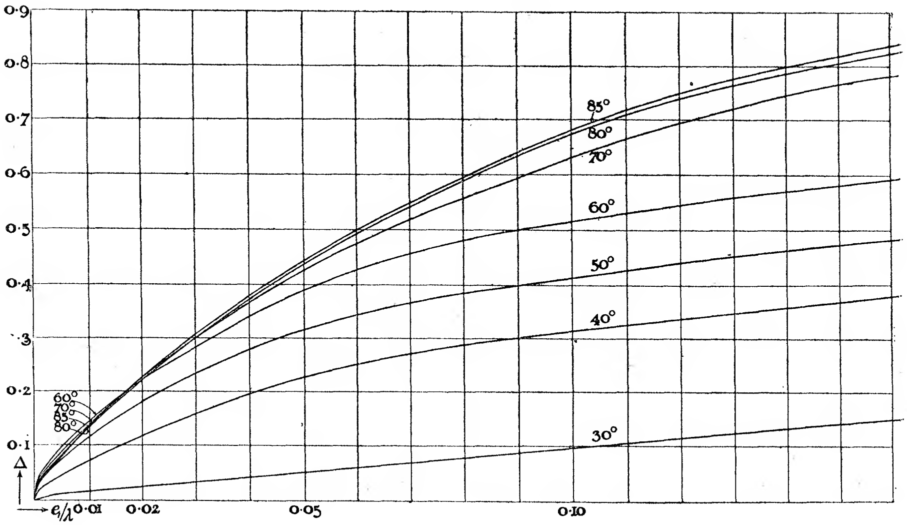


FIG. 15.

On comparing this with fig. 14 above, we see that although the general trend is the same for both metals, there are some important differences. With silver the difference of phase increases with the angle of incidence for all thicknesses. This is the case also with gold for thicknesses greater than 0·02 λ. However, for thinner films than this, Δ diminishes when ϕ is greater than 60°. With silver we have seen that for certain angles of incidence Δ increases for a time, reaches a maximum, and then slowly declines. With gold, on the other hand, Δ seems always to increase with the thickness. The increase too, is not nearly so rapid as in the case of silver, and even when $\epsilon = 0{\cdot}2\,\lambda$ we see that Δ has not yet reached a steady value.

The following table gives the values of the azimuth (ω):—

e_1/λ.	ϕ = 30°.		40°.		50°.		60°.		70°.		80°.		85°.	
	°	′	°	′	°	′	°	′	°	′	°	′	°	′
0·001	45	8	45	14	45	24	45	37	46	0	48	0	52	21
0·005	46	1	46	50	48	10	50	20	54	20	63	56	73	42
0·01	47	25	49	26	52	28	56	54	63	34	73	45	80	8
0·02	49	52	53	47	59	17	66	7	73	5	80	14	83	8
0·05	52	43	58	42	66	8	73	54	80	21	83	57	84	18
0·1	53	30	60	3	67	56	75	55	81	58	84	14	84	24
0·2	53	37	60	14	68	11	76	9	82	9	84	15	84	24

These results are represented in fig. 12 above, and on comparison with fig. 7 we see that there is very little difference between the two metals as regards the law of variation of the azimuth.

Influence of the Transition Layer.

So far we have proceeded on the hypothesis that the transition from one medium to another is abrupt and not gradual. This neglect of the layer of transition has led us to results that agree quite as well as could be expected with experiment. Unfortunately, however, the experiments that we have been able to quote are all too few in number.* They deal almost exclusively with observations of the Principal Incidence and Azimuth, and even for these the observations extend over a rather narrow range of thickness of the metallic film. We shall now turn, however, to a much more extensive series of experiments giving us the difference of phase for reflection and transmission over a wide range of thicknesses.† These experiments were made with leaves of gold and we have already (p. 318) been led to expect that for this metal the influence of the transition layer on the difference of phase would be more appreciable than with silver. When we examine Meslin's results for reflection from a *thick* plate of gold we find that the theory of an abrupt transition will not give us results that agree even approximately (except in a very rough way) with the facts of experience. It is known that on such a theory the difference of phase is given by the formula

$$\tan \Delta = \frac{2\mathrm{M}c \cos \phi \sin^2 \phi \sin (\alpha + u)}{\mathrm{M}^2 c^2 \cos^2 \phi - \sin^4 \phi},$$

and the Principal Incidence by the formula‡

$$\frac{\sin^2 \phi}{\cos \phi} = \mathrm{M}c, \quad \text{or its equivalent,} \quad \sec \phi = \mathrm{M} + \mathrm{M}^{-1}(1 - \tfrac{1}{2} \cos 2\alpha).$$

---

* It would be a great advantage if we could obtain accurate measurements of the difference of phase and the azimuth for both reflection and transmission. Meslin gives us the difference of phase, but not the azimuth.

† See note §, p. 296.

‡ See 'Roy. Soc. Proc.,' 1906, pp. 213 and 215.

Meslin found the Principal Incidence to be 74°, and as $1-\frac{1}{2}\cos 2\alpha$ must lie between 0·5 and 1·5, we see that M must lie between 3·15 and 3·49. When the incidence was 30°, Meslin found $\Delta = 11° 31'$, and from the formula for Δ we see that M cannot be greater than 2·8. This lies considerably outside the limits for M fixed by the Principal Incidence and we should be led to a similar inconsistency if we considered the value of Δ at any other incidence than that of 30°. The crosses in fig. 16 below represent Meslin's values of Δ for different incidences and it will be seen that they are very fairly consistent.* The continuous curve of the figure represents the theoretical results obtained later when the layer of transition is considered.

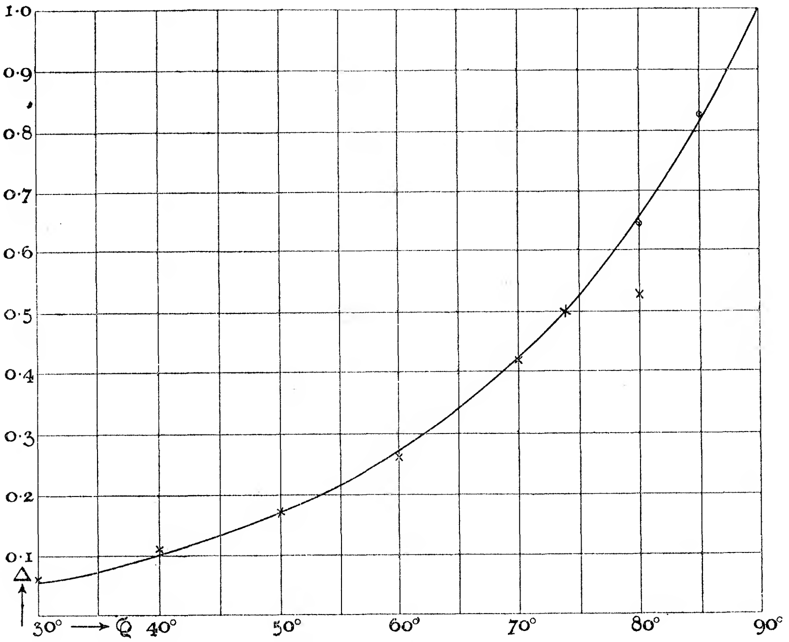


FIG. 16.

We proceed to develop some formulæ applicable to the problem when the transition layer is taken into account. In that case in addition to the optical constants, M and α, two new constants a and w are required whose values depend on the law of variation of μ within the layer. The quantities hitherto represented by r_1 and r_2 are replaced by r_1' and r_2', which are given by the following equations*:—

---

* At least up to $\phi = 74°$. There is evidently a misprint in the values of Δ given for $\phi = 80°$ and 85°. They are set down as 0·528 and 0·648 respectively. An inspection of the figure shows that these are quite inconsistent with the other results and with the graph given by Meslin. Probably the numbers should be 0·648 and 0·828, 5 being replaced by 8, and the two values interchanged. Owing to the uncertainty of these results they have been neglected altogether in the subsequent determination of the optical constants of the gold.

$$r_1' = \frac{\sin(\phi-\phi')\cos(\phi+\phi')+ae^{iw}\sin\phi'\cos\phi'\cos\phi}{\sin(\phi+\phi')\cos(\phi-\phi')+ae^{iw}\sin\phi'\cos\phi'\cos\phi}$$

$$= \frac{\mathrm{A}+a'}{\mathrm{A}'+a'}, \text{ say, } = \frac{\mathrm{A}}{\mathrm{A}'}\left[1+a'\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{A}'}\right)\right], \text{ approximately}$$

$$= r_1\left[1+a'\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{A}'}\right)\right] = r_1\left[1+p_1e^{i\lambda_1}\right],$$

where

$$p_1e^{i\lambda_1} = a'\left(\frac{1}{\mathrm{A}}-\frac{1}{\mathrm{A}'}\right) = \frac{2ae^{iw}\sin^2\phi'\cos^2\phi'\cos\phi}{\sin(\phi+\phi')\cos(\phi+\phi')\sin(\phi-\phi')\cos(\phi-\phi')}.$$

$$r_2' = -\frac{\sin(\phi-\phi')+ae^{iw}\sin\phi'}{\sin(\phi+\phi')+ae^{iw}\sin\phi'} = \frac{\mathrm{B}+b'}{\mathrm{B}'+b'} = r_2\left[1+p_2e^{i\lambda_2}\right],$$

where

$$p_2e^{i\lambda_2} = b'\left(\frac{1}{\mathrm{B}}-\frac{1}{\mathrm{B}'}\right) = \frac{2ae^{iw}\sin^2\phi'\cos\phi}{\sin(\phi+\phi')\sin(\phi-\phi')},$$

$$\frac{r_1'}{r_2'} = \frac{r_1}{r_2}(1+q_1e^{i\theta}) = s\frac{\mathrm{R}_1}{\mathrm{R}_2}e^{-i(\delta-\chi_1)},\dagger$$

$$s = (1+2q_1\cos\theta+q_1^2)^{\frac{1}{2}}, \quad \tan\chi_1 = \frac{q_1\sin\theta}{1+q_1\cos\theta}, \quad \theta = 2\alpha+w+\theta',$$

$$q_1 = \frac{2a\sin^2\phi\cos\phi}{[\mathrm{M}^4\cos^4\phi+c^4-2\mathrm{M}^2c^2\cos^2\phi\cos 2(\alpha-u)]^{\frac{1}{2}}};$$

$$\tan\theta' = \frac{c^2\sin 2(\alpha-u)}{\mathrm{M}^2\cos^2\phi-c^2\cos 2(\alpha-u)}.$$

δ is the difference of phase that an abrupt transition would produce, and is given by the formula for Δ on p. 325, while χ is the correction due to the transition layer.

Since q_1 and θ must be calculated in order to determine χ_1, it will be convenient to express p and λ in terms of these quantities. From the above formulæ we have

$$p_1e^{i\lambda_1} = q_1e^{i\theta}\frac{\cos^2\phi'}{\sin^2\phi} = q_1\frac{c^2e^{i(\theta-2u)}}{\sin^2\phi},$$

whence

$$p_1 = q_1c^2/\sin^2\phi \quad \text{and} \quad \lambda_1 = \theta-2u.$$

Similarly

$$p_2e^{i\lambda_2} = q_1e^{i\theta}\frac{\cos(\phi+\phi')\cos(\phi-\phi')}{\sin^2\phi} = \frac{q_1e^{i\theta}}{\sin^2\phi}[\cos^2\phi'-\sin^2\phi]$$

$$= \frac{q_1e^{i\theta}}{\sin^2\phi}[c^2e^{-i2u}-\sin^2\phi],$$

---

* See 'Roy. Soc. Proc.,' 1906, p. 222.

† *Ibid.*, p. 222—3. χ has been replaced by χ_1 and q by q_1 to avoid confusion with the quantity called q in the earlier part of this paper. The quantity q is printed with the wrong sign in the paper referred to, and θ and θ' of p. 224 should each be increased by 180°.

whence

$$p_2 = \frac{q_1}{\sin^2\phi}(c^4+\sin^4\phi-2c^2\sin^2\phi\cos 2u)^{\frac{1}{2}} = \frac{q_1 c^2}{\sin^2\phi}\cdot\frac{\sin 2u}{\sin\Omega} = p_1\frac{\sin 2u}{\sin\Omega},$$

and $$\lambda^2 = \theta-\Omega,$$

where $$\tan\Omega = \frac{c^2\sin 2u}{c^2\cos 2u-\sin^2\phi}.$$

It thus appears that there will be no great difficulty in calculating all the quantities involved in our formulæ, once the four optical constants M, α, a, and w are obtained. However, to obtain these four quantities *accurately* from a series of observations of the difference of phase ($\delta - \chi_1$) at different angles of incidence is a problem involving some labour. Theoretically, we could obtain the four constants from any four observations of $\delta - \chi_1$; but the equations are much too complicated to be solved as a set of four simultaneous equations. Practically we must proceed by a series of approximations and in this way consistent results are soon obtained, unless very great accuracy is aimed at. From the formulæ it appears that χ_1 (the correction to the change of phase due to the layer) vanishes at normal incidence and is small when the angle of incidence is small. We can, therefore, obtain a first approximation to M and α from two observations of the difference of phase when the angle of incidence is not large. As a first approximation in such circumstances we can neglect χ_1 altogether, or give to it any small value that seems reasonable. Knowing M and α (approximately) we can calculate δ, q_1/a, and θ' from the formulæ above. Having obtained δ by calculation, and $\delta - \chi_1$ from observation for any other angle of incidence, we thus obtain χ_1 for that incidence. Also we have

$$\tan\chi_1 = \frac{q_1\sin\theta}{1+q_1\cos\theta},$$

so that $$\frac{1}{a} = \frac{q_1}{a}\frac{\sin(\theta-\chi_1)}{\sin\chi_1} = \frac{q_1}{a}\frac{\sin(w+2\alpha+\theta'-\chi_1)}{\sin\chi_1}.$$

In this equation everything is known (approximately) except a and w, so that by using our knowledge of $\delta - \chi$ at two angles of incidence, other than those employed for obtaining the approximate values of M and α, we easily obtain the unknown quantities a and w. Having thus obtained approximate values of the four optical constants, we can calculate χ in terms of them, and proceed, if necessary, to higher approximations. If the experimental results give us the difference of phase for more than four angles of incidence, we can use the ordinary rules for finding the most probable values of the optical constants, and estimate the probable errors.

In carrying out the process sketched above with reference to any set of observations actually available at present, we find that the approximations need not be carried very far, for we soon obtain results that fit in with experiment well within the limits of experimental errors. We take Meslin's results with red light reflected from gold; and the second approximation in the above process yields the following values of the optical constants:— $M = 2·63$; $\alpha = 70°$; $a = 0·7$; $w = 70° \; 39'$. From these we derive the following table, giving the values of χ, $\Delta \, (= \delta - \chi_1)$, and the difference between theory and observation:—

ϕ.	30°.	40°.	50°.	60°.	70°.	74°.	80°.	85°.
χ_1	0·009	0·017	0·027	0·041	0·055	0·061	0·051	0·027
Δ (calculated) ...	0·056	0·103	0·176	0·270	0·423	0·5	0·663	0·815
Δ (observed)......	0·064	0·112	0·176	0·260	0·420	0·5	0·648	0·828
Difference ...	−0·008	−0·009	0	+0·010	+0·003	0	+0·015	−0·013

These results are represented in fig. 16 above, in which the continuous curve corresponds to the theory and the crosses to the experiment. From the figure or the table we see that the agreement between theory and experiment is a close one, and that, in view of the uncertainty as to some of the experimental results, not much could be gained by proceeding to a higher approximation. We shall therefore take the constants given above as the basis of our calculations. From the formulæ already given we then derive the following table:—

ϕ.	30°.	40°.	50°.	60°.	70°.	80°.
u	0° 38′	1° 3′	1° 25′	1° 50′	2° 8′	2° 20′
c	1·014	1·023	1·032	1·042	1·047	1·054
$180-\theta_1$	46° 5′	51° 56′	61° 4′	75° 46′	99° 28′	135° 23′
$180-\theta_2$	34° 25′	30° 22′	25° 22′	19° 44′	13° 29′	6° 50′
R_1^2	0·5861	0·5505	0·5105	0·3720	0·3857	0·4196
R_2^2	0·6738	0·7079	0·7504	0·7834	0·8602	0·9270
m	2·667	2·690	2·716	2·742	2·760	2·773
χ	70° 39′	71° 3′	71° 28′	71° 50′	72° 8′	72° 20′
ψ	63° 36′	62° 55′	62° 10′	61° 32′	60° 58′	60° 37′
Log Q	$\bar{2}$·626	$\bar{2}$·610	$\bar{2}$·594	$\bar{2}$·578	$\bar{2}$·566	$\bar{2}$·558
$\lambda_1-360°$	35° 47′	36° 49′	39° 5′	43° 34′	52° 4′	63° 24′
$\lambda_2-360°$	35° 23′	35° 27′	35° 38′	35° 26′	35° 38′	33° 14′
p_1	0·2076	0·2297	0·2606	0·2894	0·2974	0·2830
p_2	0·1578	0·1392	0·1177	0·0905	0·0621	0·0401

We are now in a position to discuss all the problems dealt with in the earlier part of this paper without neglecting the layer of transition from one medium to another. The effect of this layer is to replace by $r(1 + pe^{i\lambda})$. The other quantities remain as before, but it must be remembered that q on pp. 297 *et seq.* refers to a thickness t_1 of *homogeneous* metal. We shall set out our results in terms of t_1, but, of course, this is not the actual thickness of the plate. To get that thickness we must increase t_1 by twice the thickness of the layer, and it is owing to the uncertainty as to the value of this latter quantity that we cannot state the results precisely in terms of the thickness of the metallic plate. If d be the thickness of the layer we should expect* $2\pi d/\lambda$ to lie between a/M^4 and a/M^2, and with the constants given above this makes d/λ lie between 0·0023 and 0·0161.†

If light be incident normally on a gold leaf whose optical constants are those just determined, then the intensity and change of phase for the reflected and transmitted light are as follows :—

e_1/λ.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.
R'^2 (reflection) ……	0·001	0·023	0·079	0·234	0·660	0·870	0·861
R'^2 (transmission)…	0·933	0·845	0·736	0·547	0·218	0·049	0·002
θ' (reflection) ……	0·655	0·678	0·700	0·739	0·791	0·815	0·800
θ' (transmission) …	0·001	0·013	0·056	0·079	0·106	0·068	−0·101

These results are represented graphically in Curve 3 of figs. 1, 2, 3 and 4, above.

If the gold, instead of being surrounded by air, is deposited on glass, these results must be modified as in the case of silver on p. 305. The intensity and change of phase for the reflected light are then given by the following table :—

e_1/λ.	0·001.	0·005.	0·01.	0·02.	0·05.	0·1.	0·2.
R_1^2 ………	0·035	0·030	0·077	0·219	0·644	0·860	0·860
θ' …………	0·081	0·307	0·509	0·662	0·776	0·815	0·800

---

* See 'Roy. Soc. Proc.,' 1906, p. 229. With the law of variation of μ^2 given on p. 230, we have, for the gold alone, $E_1 = 2·228$. This makes $d/\lambda = 0·007$, which is about half way between the limits found in the text, and is almost identical with the value obtained for the layer of transition in the case of reflection from diamond. See 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 63.

† That so thin a layer should produce an appreciable change in the Principal Incidence and Azimuth is not altogether surprising after what has been seen on p. 315. The gold under discussion here is a good example of the error that may be made in deriving the optical constants of a metal from observations of the Principal Incidence and Azimuth and neglect of the transition layer. The Principal Incidence is 74° and the Principal Azimuth, obtained from the formula $\tan\beta = R_1/R_2 . (1+2q_1 \cos\theta+q_1^2)^{\frac{1}{2}}$, is 44° 57′. Neglecting the layer of transition these would give $M = 3·316$ instead of $M = 2·63$ and $a = 87° 40'$ instead of $a = 78°$. If $Me^{-ia} = \mu_1 - ia$, then we should get $\mu_1 = 0·135$, instead of $\mu_1 = 0·9$ and $a_1 = 3·31$ instead of $a_1 = 2·47$.

These results are exhibited in Curve 6 of figs. 1 and 2 above. From the first of these figures we see, by comparing Curves 1 and 4, or 2 and 5, that for silver or gold and an abrupt transition the intensity of the reflected light is always greater for the metal deposited on glass than for the metal plate alone. The difference between the two cases is, however, small, and tends to zero as the thickness increases. On comparing Curves 3 and 6 we see that in this respect the layer of transition modifies the results considerably. After a thickness $t_1 = 0{\cdot}001\,\lambda$ has been reached, the intensity of the light reflected from gold on glass is *less* than that reflected from gold alone.

To determine the ellipticity difference of phase when the light is incident at any angle ϕ we have to modify the formulæ on p. 308 by replacing r by $r' = r(1+pe^{i\lambda})$. For the reflected light we have

$$\frac{R_1'}{R_2'}e^{i(\theta_1'-\theta_2')} = \frac{r_1'}{r_2'}\frac{(1-q_1^2r_2'^2)}{(1-q_1^2r_1'^2)},$$

where

$$\frac{r_1'}{r_2'} = \frac{r_1}{r_2}(1+q_1e^{i\theta}) = s\frac{R_1}{R_2}e^{-i(\delta-\chi_1)} \quad \text{and} \quad s = (1+2q_1\cos\theta+q_1^2)^{\frac{1}{2}}.$$

$$1-q_1^2r'^2 = 1-QR^2e^{i(2\theta-\psi)}(1+2pe^{i\lambda}) \text{ approximately,}$$

$$= [1-QR^2\{\cos(\psi+\gamma)+2p\cos(\psi+\gamma-\lambda)\}] + iQR^2\{\sin(\psi+\gamma)+2p\sin(\psi+\gamma-\lambda)\},$$

where $\gamma = 360°-2\theta$.

Thus $1-q_1^2r'^2 = Ae^{i\Phi}$, where A and Φ are the modulus and argument respectively of the complex quantity.

$$1-QR^2\{\cos(\psi+\gamma)+2p\cos(\psi+\gamma-\lambda)\}+iQR^2\{\sin(\psi+\gamma)+2p\sin(\psi+\gamma-\lambda)\}.$$

The difference of phase is

$$\Delta = \theta_2'-\theta_1' = \delta-\chi_1+\Phi_1-\Phi_2,$$

and the ellipticity is

$$\epsilon = \tan\omega = \frac{sR_2}{R_1}\frac{A_2}{A_1}.$$

From these formulæ we get the following table for Δ:—

e_1/λ.	ϕ = 30°.	40°.	50°.	60°.	70°.	80°.
0·001	0·112	0·205	0·341	0·524	0·795	1·195
0·005	0·103	0·188	0·311	0·477	0·718	1·052
0·01	0·094	0·170	0·283	0·433	0·652	0·965
0·02	0·081	0·147	0·246	0·380	0·579	0·865
0·05	0·067	0·127	0·206	0·317	0·492	0·743
0·1	0·060	0·109	0·185	0·283	0·440	0·679
0·2	0·057	0·103	0·176	0·271	0·424	0·664

These results are represented in fig. 17 below.

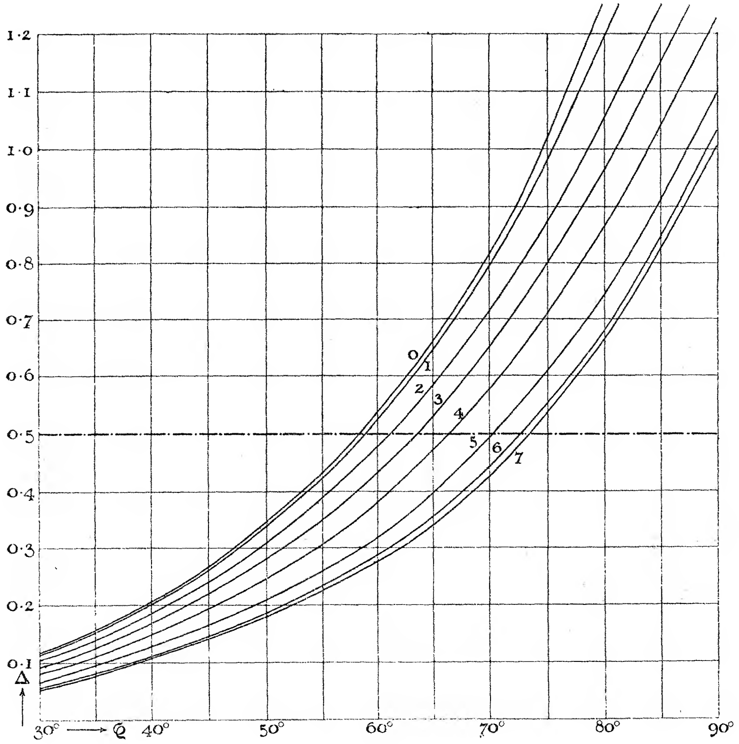


FIG. 17.

A comparison of this figure with fig. 11 above will indicate the nature of the modifications introduced by the layer. As far as the variation of Δ with change of ϕ for any given thickness is concerned, there is not any very marked difference between the two figures. The main difference lies in the variation due to change of thickness. When the layer is taken into account, Δ always diminishes with the thickness; but without the layer this is not always the case. It will be seen, too, that Curves 5, 6, 7 of fig. 11 are much more crowded than the corresponding curves of fig. 17. Without the layer there is very little change in Δ for thicknesses greater than 0·05 λ; but with the layer there is still a considerable change after ϵ_1 has passed that value.

The Principal Incidence corresponding to each thickness is obtained from the graph. It increases steadily with the thickness, its values being 58° 20′, 58° 48′, 61° 10′, 63° 40′, 66° 18′, 70° 15′, 72° 40′, and 73° 30′.

If the incident light be of unit intensity and be polarised in a plane making an angle of 45° with the plane of incidence, then the intensity of the reflected light is given by the formula

$$I = \tfrac{1}{2}(R_1'^2 + R_2'^2).$$

We have $r' = r(1+pe^{i\lambda}) = ae^{ia}$,

where $a = (1+2p\cos\lambda+p^2)^{\frac{1}{2}}$ and $\tan\alpha = p\sin\lambda/(1+p\cos\lambda)$.

$$1-q^2r'^2 = \mathrm{A}e^{i\Phi};\quad 1-q^2 = 1-\mathrm{Q}e^{-i\psi} = be^{i\beta},$$

where $b = (1-2\,\mathrm{Q}\cos\psi+\mathrm{Q}^2)^{\frac{1}{2}}$ and $\tan\beta = \mathrm{Q}\sin\psi/(1-\mathrm{Q}\cos\psi)$.

Whence $$\mathrm{R}'^{i\theta'} = \frac{r'(1-q^2)}{1-q^2r'^2} = \frac{ab\mathrm{R}}{\mathrm{A}}e^{i[\theta+\alpha+\beta-\Phi]}.$$

Thus $\mathrm{R}' = ab\mathrm{R}/\mathrm{A}$ and $\theta' = \theta+\alpha+\beta-\Phi$.

From these formulæ we derive the following values of I:—

$e_1/\lambda =$	0·001	0·005	0·01	0·02	0·05	0·1	0·2
30°	0·001	0·017	0·065	0·192	0·528	0·773	0·836
40	0·001	0·018	0·071	0·203	0·537	0·772	0·845
50	0·001	0·026	0·083	0·223	0·547	0·770	0·829
60	0·002	0·035	0·109	0·252	0·570	0·760	0·783
70	0·005	0·063	0·162	0·325	0·604	0·800	0·838
80	0·014	0·171	0·295	0·457	0·665	0·817	0·857

These results are represented in fig. 18 below.

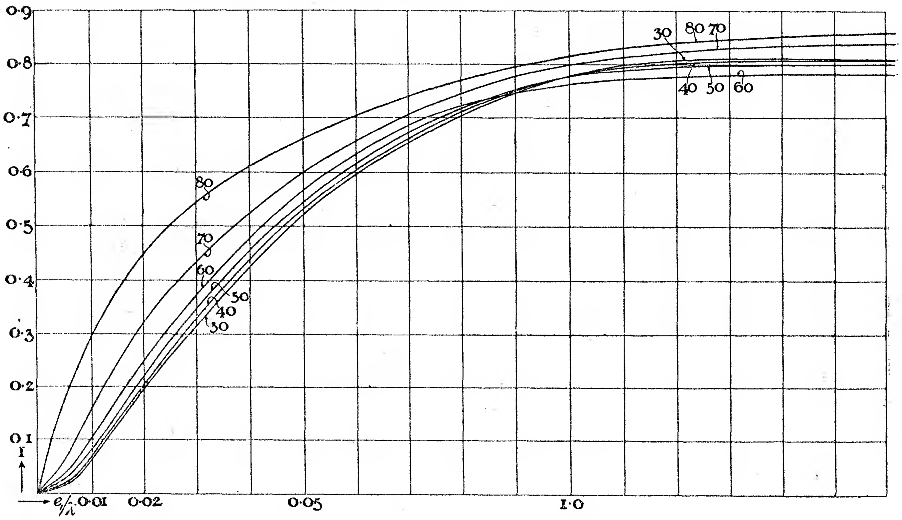


FIG. 18.

For thicknesses less than 0·08 λ the intensity increases with the angle of incidence, but for greater thicknesses the intensity obeys a different law. The law approaches more and more closely to that for a thick plate, the intensity diminishing slightly until the "quasi-polarising" angle is reached and then increasing towards unity as the angle of incidence tends towards 90°

For the Azimuth (ω) we get the following set of values:—

e_1/λ.	ϕ = 30°.		40°.		50°.		60°.		70°.		80°.	
	°	′	°	′	°	′	°	′	°	′	°	′
0·001	37	46	32	5	25	10	17	34	11	25	6	38
0·005	38	14	32	55	26	25	19	5	13	17	9	16
0·01	38	55	34	7	28	11	21	16	16	7	13	48
0·02	40	11	36	24	31	42	26	44	21	45	20	54
0·05	42	41	40	49	38	31	35	10	32	16	32	28
0·1	43	51	43	41	42	0	39	21	37	15	36	3
0·2	44	7	43	24	42	30	39	41	37	35	36	4

These results are represented in fig. 19 below, in which the crosses indicate the position of the Principal Incidence and the Principal Azimuth.

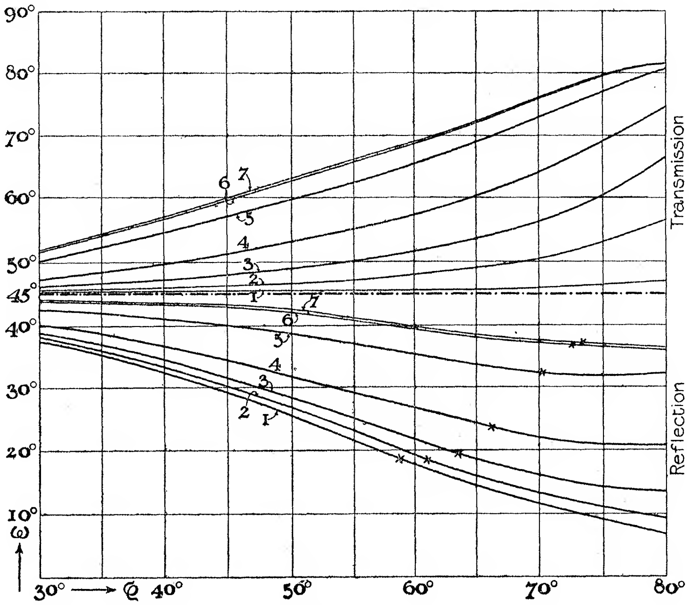


Fig. 19.

A comparison with fig. 12 will show that the influence of the layer of transition on the law of variation of the ellipticity is very slight.

Turning to the transmitted beam, we proceed as on pp. 321 and 322, and find that the difference of phase and the ellipticity are given by the formulæ

$$\Delta = \Phi_1 - \Phi_2 - (\Phi_1' - \Phi_2') \quad \text{and} \quad \epsilon = \tan\omega = \frac{A_2}{A_1}\frac{A_1'}{A_2'},$$

where A and Φ have the same values as on p. 331, while A' and Φ' are derived from them by putting $Q = 1$ and $\psi = 0$.

From these we get the following table for Δ :—

e_1/λ.	ϕ = 30°.	40°.	50°.	60°.	70°.	80°.
0·001	0·004	0·005	0·008	0·013	0·025	0·052
0·005	0·013	0·022	0·038	0·060	0·102	0·195
0·01	0·023	0·040	0·067	0·104	0·167	0·282
0·02	0·035	0·063	0·103	0·157	0·241	0·382
0·05	0·049	0·083	0·143	0·220	0·328	0·503
0·1	0·057	0·101	0·165	0·254	0·379	0·567
0·2	0·059	0·107	0·173	0·267	0·395	0·583

Fig. 20 below represents these results graphically.

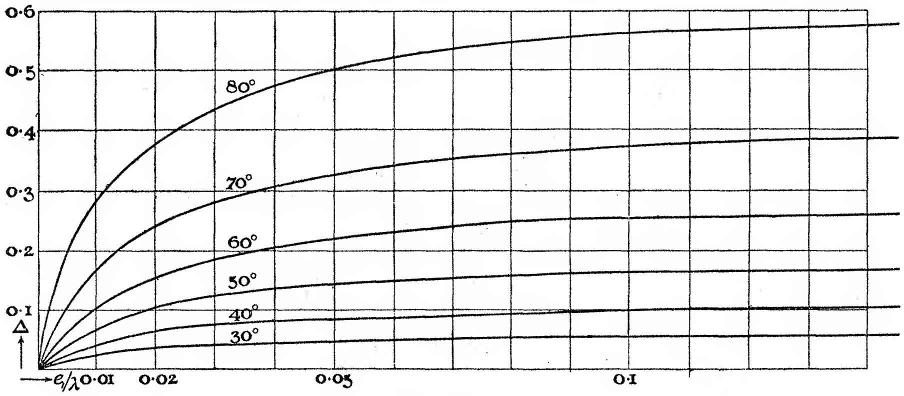


FIG. 20.

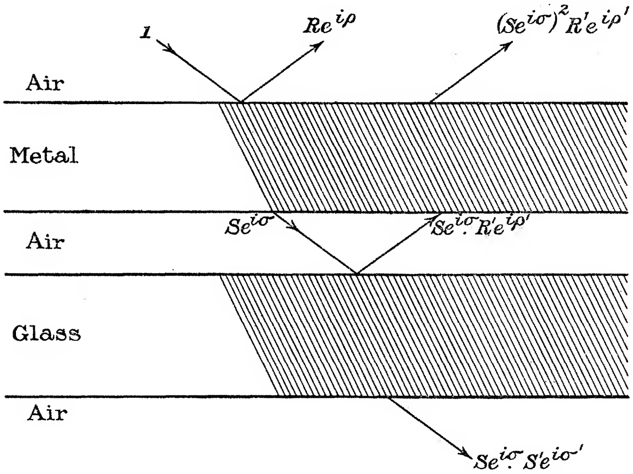
A comparison with fig. 15 shows that the layer of transition modifies the law of variation of Δ to some extent. The curves in fig. 20 correspond more closely to those for silver in fig. 14 than to those for gold in 15. The difference of phase always increases with the incidence and as the thickness increases it soon reaches a steady value. This was not the case with homogeneous gold (see p. 324). It should be noted, too, that though the principal optical constants M and α are not very different for the two specimens of gold that have been considered, the numerical values of Δ are appreciably different in the two cases. Hence to get a close numerical agreement between theory and experiment the optical constants would have to be determined accurately.

The azimuth is given in the table on p. 336.

Fig. 19 above represents these results and a comparison with fig. 12 will show that the azimuth is little affected by the layer of transition—as we found was the case with the reflected beam also.

Before we can compare these results with Meslin's experiments we must ascertain to what extent they are modified when the gold is deposited on glass instead of being surrounded by air. This can be done most simply by the method described on p. 306 above.

e_1/λ.	ϕ = 30°.		40°.		50°.		60°.		70°.		80°.	
	°	′	°	′	°	′	°	′	°	′	°	′
0·001	45	6	45	10	45	20	45	35	45	58	46	57
0·005	45	34	46	5	46	56	48	6	50	22	56	21
0·01	46	16	47	23	49	6	51	26	55	55	66	8
0·02	47	34	49	47	53	4	57	2	63	52	74	8
0·05	50	4	54	11	59	45	65	10	72	48	80	18
0·1	51	13	56	18	62	43	68	14	75	35	81	31
0·2	51	29	56	37	63	8	68	33	75	45	81	32


I
$Re^{i\rho}$
$(Se^{i\sigma})^2R'e^{i\rho'}$
Air
Metal
$Se^{i\sigma}$
$Se^{i\sigma}R'e^{i\rho'}$
Air
Glass
Air
$Se^{i\sigma}S'e^{i\sigma'}$

A beam of unit amplitude incident on the metal gives rise to a reflected beam $Re^{i\rho}$ and a transmitted beam $Se^{i\sigma}$, where, as we have seen,

$$Re^{i\rho} = \frac{r(1-q^2)}{1-q^2r^2} \quad \text{and} \quad Se^{i\sigma} = \frac{q(1-r^2)}{1-q^2r^2}.$$

The beam $Se^{i\sigma}$ is then incident on a glass plate and gives rise to a reflected beam $Se^{i\sigma}.R'e^{i\rho'}$ and a transmitted beam $Se^{i\sigma}.S'e^{i\sigma'}$. Here

$$R'e^{i\rho'} = \frac{r'(1-q'^2)}{1-q'^2r^2} \quad \text{and} \quad S'e^{i\sigma'} = \frac{q'(1-r'^2)}{1-q'^2r'^2},$$

where r' is obtained from Fresnel's formulæ for reflection and $q' = e^{-i2\pi\lambda^{-1}\mu'\cos\phi'.e'}$, where μ' is the coefficient of reflection and e' is thickness of the glass.

The wave $Se^{i\sigma}R'e^{i\rho}$ reflected from the glass is transmitted through the metal plate and gives rise to a wave $(Se^{i\sigma})^2R'e^{i\rho'}$ in the air. Thus the transmitted wave is $SS'e^{i(\sigma+\sigma')}$ and the reflected wave is the resultant of $Re^{i\rho}$ and $(Se^{i\sigma})^2R'e^{i\rho'}$.

These results are quite general and would suffice to solve the problem in the most complicated case when both plates (metal and glass) were thin, *i.e.*, not large compared with a wave-length. In the problem at present in hand the glass plate is thick, and under these circumstances there is a considerable simplification. We then have $S'e^{i\sigma'} = q'$ and $R'e^{i\rho'} = r'$, so that the transmitted beam is $q'Se^{i\sigma}$ and the reflected one is the resultant of $Re^{i\rho}$ and $r'(Se^{i\sigma})^2$.

The transmitted wave is most readily disposed of. The effect of the glass is to replace $Se^{i\sigma}$ by $q'Se^{i\sigma}$. The factor q' is the same whether the light be polarised parallel or perpendicularly to the plane of incidence. Hence the glass has no influence on the *difference* of phase nor on the ellipticity of the transmitted beam. We should expect, then, that the results on p. 334 above should agree, within the limits of experimental error, with Meslin's values. A comparison shows that there is a very fair agreement as far as the law of variation of Δ is concerned. Owing to the uncertainty as to the thickness of the layer of transition and the difficulty of obtaining accurate measurements of the thickness of the metallic deposit, we cannot compare the numerical results of theory and experiment with much hope of an exact equivalence. As we shall see later, the experiments on reflection, when compared with theory, give us a fair basis of comparison as far as thickness is concerned. On determining in this way the thicknesses to be compared, it appears that the theoretical results, although obeying much the same law of dependence on ϕ as Meslin's values, are uniformly higher than these values. However, as was observed on p. 335 above, a slight change in the optical constants makes an appreciable difference in the difference of phase of the transmitted light—and in view of this fact the agreement between theory and experiment is as close as could be expected. Unfortunately, Meslin made no record of the ellipticity, so that we have no means of comparison in this region.

The reflected light is the resultant of $Re^{i\rho}$ and $r'(Se^{i\sigma})^2$. Denoting this by $R'e^{i\theta'}$, we have

$$R'^2 = R^2 + r'^2S^4 + 2r'RS^2(\rho - 2\sigma),$$

and

$$\tan\theta' = \frac{R\sin\rho + r'S^2\sin 2\sigma}{R\cos\rho + r'S^2\cos 2\sigma}.$$

The difference of phase Δ is given by

$$\Delta = \theta_2' - \theta_1',$$

and the ellipticity (ϵ) by

$$\epsilon = \tan\omega = R_1'/R_2'.$$

The quantities R and ρ of these formulæ are identical with R′ and θ' of p. 333; *i.e.*, $\mathrm{R} = ab\mathrm{R}/\mathrm{A}$ and $\rho = \theta+\alpha+\beta-\Phi$. To determine S and σ we have

$$\mathrm{S}e^{i\sigma} = \frac{q(1-r^2)}{1-q^2r'^2} = \frac{\mathrm{Q}^{\frac{1}{2}}e^{-\frac{1}{2}i\psi}\mathrm{A}'e^{i\Phi'}}{\mathrm{A}e^{i\Phi}}.$$

Hence $$\mathrm{S}^2 = \mathrm{Q}(\mathrm{A}'/\mathrm{A})^2 \quad\text{and}\quad \sigma = \Phi'-\Phi-\tfrac{1}{2}\psi.$$

The only other quantity involved is r'. This is given by Fresnel's formulæ. This, of course, involves the index of refraction of the glass, and this, unfortunately, Meslin did not put on record. In our calculations we shall take $\mu' = 1·54$, which corresponds to a polarising angle of 57°.

The following table gives the values of R_1':—

	$e_1/\lambda = 0·01$.	0·02.	0·05.	0·1.	0·2.
°					
30	0·221	0·385	0·687	0·854	0·899
40	0·200	0·362	0·667	0·843	0·887
50	0·179	0·338	0·638	0·812	0·857
60	0·180	0·316	0·573	0·723	0·749
70	0·278	0·379	0·618	0·743	0·750
80	0·529	0·599	0·756	0·842	0·850

These results are represented in fig. 21 below.

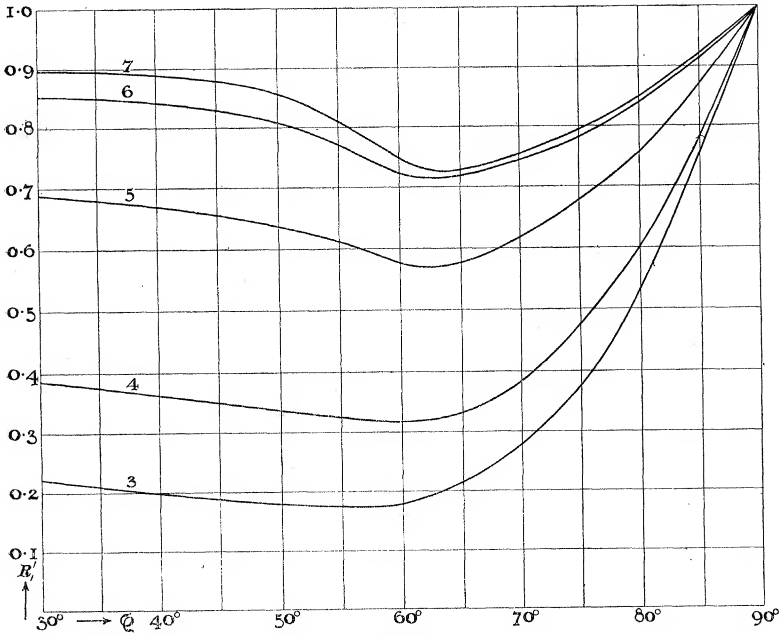


FIG. 21.

R_1' increases rapidly with the thickness. It diminishes as the incidence increases, reaches a minimum, and then increases to unity. The "quasi-polarising" angle—*i.e.*, the angle of incidence that makes R_1' a minimum—increases slowly with the thickness. For the thicknesses represented in the above table and figure this quasi-polarising angle has the values 58°, 60°, 62°, 63°, and 64°.

The azimuth (ω) is given in the following table:—

e_1/λ =	0·01.		0·02.		0·05.		0·1.		0·2.	
°	°	′	°	′	°	′	°	′	°	′
30	38	31	39	7	42	14	43	37	44	3
40	32	59	34	53	40	22	42	52	43	22
50	25	42	30	8	37	43	41	0	41	50
60	21	2	25	40	33	52	37	50	38	16
70	24	36	26	17	34	11	37	48	37	56
80	33	49	34	15	37	58	38	1	38	58

Fig. 22 below represents these values.

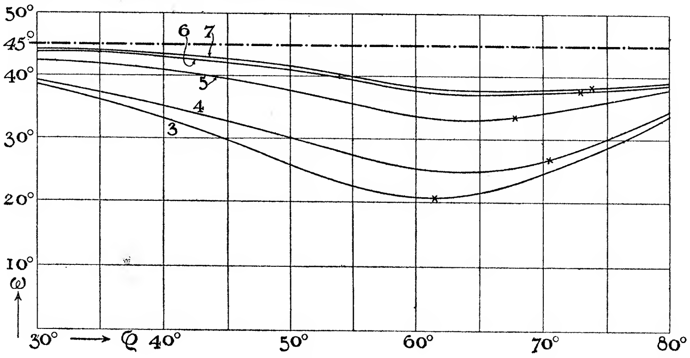


FIG. 22.

The crosses indicate the position of the Principal Azimuth, which increases through the following values: 21°, 26° 45′, 33° 30′, 37°, 37° 30′.

Finally, the phase difference (Δ) has these values:—

	0·01.	0·02.	0·05.	0·1.	0·2.
°					
30	0·008	0·033	0·046	0·055	0·058
40	0·028	0·086	0·105	0·104	0·105
50	0·079	0·190	0·214	0·190	0·178
60	0·483	0·317	0·335	0·280	0·273
70	0·887	0·493	0·525	0·440	0·428
80	0·990	0·755	0·805	0·695	0·665

These results are represented in fig. 23 below.

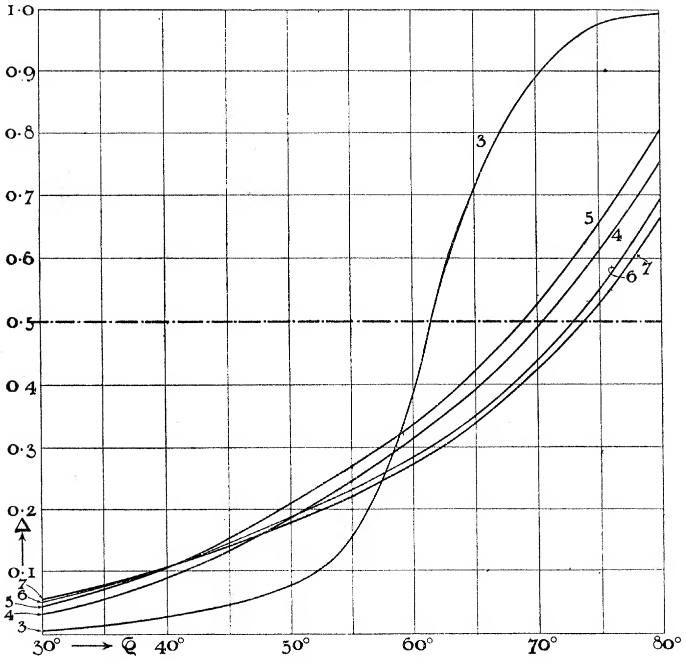


Fig. 23.

A comparison with fig. 17 above will show that the glass modifies the results in a very marked manner where the thickness of the metal is small. The difference of phase increases with the thickness for angles of incidence less than 40°. For larger angles, however, the law is completely different. When ϕ is greater than 60° the difference of phase at first diminishes rapidly with the thickness, then increases for a while, and afterwards diminishes again till it reaches the value appropriate to a thick plate. The Curve (3) corresponding to $e_1/\lambda = 0{\cdot}01$ has a point of inflexion and cuts all the other curves. As is to be expected, its form is very similar to that of the curve representing the difference of phase in the case of reflection from glass or any other transparent medium when account is taken of the layer of transition.* The Principal Incidence, instead of increasing steadily with the thickness, as is the case with gold alone, increases rapidly at first, diminishes for a time and then increases again. Its values for the thicknesses here discussed are 61° 30′, 70° 30′, 67° 45′, 73° and 74°.

The graphical representation of Meslin's results in fig. 24 below will facilitate a comparison between theory and experiment.

* See 'Roy. Soc. Proc.,' 1905, p. 64.

On comparing this with fig. 23 above we see that the agreement is a close one. Curves *a*, *b*, and *c* of fig. 24 correspond to the following thicknesses as found by Meslin, 6 $\mu\mu$, 29 $\mu\mu$, and 41 $\mu\mu$; while *d* corresponds to Meslin's values for a thick plate. The Curve *d* is practically identical with the Curve 7 of fig. 23, while the Curves *c*, *b*, and *a* are very similar to Curves 5, 4, and 3 respectively. Taking $\lambda = 760\ \mu\mu$, the thickness of the metallic film corresponding to the Curves *a*, *b*, and *c* is $0{\cdot}008\,\lambda$, $0{\cdot}038\,\lambda$, and $0{\cdot}054\,\lambda$ respectively. Comparing these numbers with the values of ϵ_1, corresponding to the Curves 3, 4, and 5, we see that the thickness of the layer of transition is of the order assigned to it on p. 330 above.*

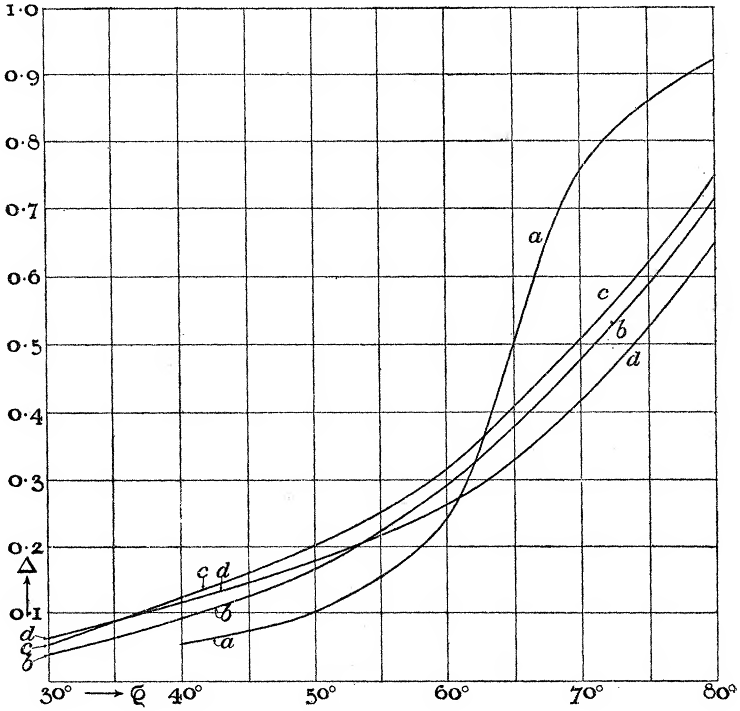


Fig. 24.

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* The surface conditions of the various films and of the thick plate were probably different, so that the layer of transition would not be the same in each. These results, however, indicate that the thickness of the layer was of the same order in each case. The uncertainty as to the refractive index of the glass has most influence on the thinnest film.